Arnold diffusion for dummies

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October 4^{th}, 2017

In 1964, V.I. Arnold proposed an example of a nearly-integrable Hamiltonian with $2+1/2$ degreees of freedom

$$H(q,p,\varphi,I,t)=\frac{1}{2}\left(p^2+I^2\right)+\varepsilon(\cos q-1)\left(1+\mu(\sin\varphi+\cos t)\right),$$

and asserted that given any $\delta, K>0$, for any $0<\mu\ll\varepsilon\ll 0$, there exists a trajectory of this Hamiltonian system such that

$$I(0)<\delta \text{ and } I(T)>K \qquad \text{for some time } T>0.$$

Notice that this a global instability result for the variable I, since

$$\dot{I}=-\frac{\partial H}{\partial\varphi}=-\varepsilon\mu(\cos q-1)\cos\varphi$$

is zero for $\varepsilon=0$, so I remains constant, whereas I can have a drift of finite size for *any* $\varepsilon>0$ small enough.

Arnold's Hamiltonian can be written as a nearly-integrable with 3 degrees of freedom

$$H^*(q, p, \varphi, I, s, A) = \frac{1}{2}\left(p^2 + I^2\right) + A + \varepsilon(\cos q - 1)\left(1 + \mu(\sin\varphi + \cos s)\right),$$

which for $\varepsilon = 0$ is an integrable Hamiltonian $h(p, I, A) = \frac{1}{2}\left(p^2 + I^2\right) + A$. Since h satisfies the (Arnold) *isoenergetic nondegeneracy*

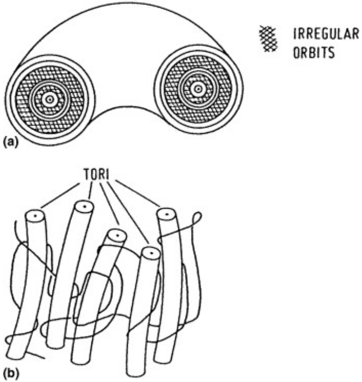
$$\begin{vmatrix} D^2 h & Dh \\ Dh^\top & 0 \end{vmatrix} = -1 \neq 0$$

By the KAM theorem proven by Arnold in 1963, the 5D phase space of H is filled, up to a set of relative measure $\mathrm{O}(\sqrt{\varepsilon})$, with 3D-invariant tori $\mathcal{T}_\omega$ with Diophantine frequencies $\omega = (\omega_1, \omega_2, 1)$:

$$|k_1\omega_1 + k_2\omega_2 + k_0| \geq \gamma/|k|^\tau \text{ for any } 0 \neq (k_1, k_2, k_0) \in \mathbb{Z},$$

where $\gamma = \mathrm{O}(\sqrt{\varepsilon})$, and $\tau \geq 2$.

Figure: a): 2D tori separate a 3D phase space. b) 3D tori do not separate a 5D phase space



Since the 3D KAM invariant tori do not separate the 5D phase space, there can exist irregular orbits 'traveling' between tori. Arnold conjectured in the KAM theorem in 1963 that this was the general case.

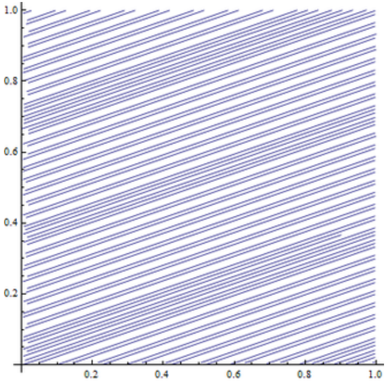
The unperturbed rôle is played by a (completely) integrable Hamiltonian with n degrees of freedom. The Liouville–Arnold theorem establishes, under certain hypotheses, the existence on some region of the phase space of canonical *action–angle variables* $(\varphi, I) = (\varphi_1, \dots, \varphi_n, I_1, \dots, I_n)$ in $\mathbb{T}^n \times G \subset \mathbb{T}^n \times \mathbb{R}^n$, in which the Hamiltonian only depends on the action variables: $h(I)$. The associated Hamiltonian equations for a trajectory $(\varphi(t), I(t))$ are

$$\dot{\varphi} = \omega(I), \qquad \dot{I} = 0,$$

where $\omega = \partial_I h$. Hence the dynamics is very simple: every n-dimensional torus $I =$ constant is invariant, with linear flow $\varphi(t) = \varphi(0) + \omega(I)t$, and thus all trajectories are stable. The motion on a torus is called quasiperiodic, with associated *frequencies* given by the vector $\omega(I) = (\omega_1(I), \dots, \omega_n(I))$.

Every n-dimensional invariant torus can be non-resonant or resonant, according to whether its frequencies are rationally independent or not. A non-resonant torus is densely filled by any of its trajectories. On the other hand, a resonant torus is foliated into a family of lower dimensional tori.

Figure: Non-resonant 2D Torus



A *nearly-integrable* Hamiltonian can be written in the form

$$H(\varphi, I) = h(I) + \varepsilon f(\varphi, I), \tag{1}$$

where ε is a small perturbation parameter. Then the Hamiltonian equations are

$$\dot{\varphi} = \omega(I) + \varepsilon \partial_I f(\varphi, I), \qquad \dot{I} = -\varepsilon \partial_\varphi f(\varphi, I).$$

For non-resonant, even more, Diophantine frequencies, KAM theorem provides n-dimensional invariant tori. For resonant frequencies there appear, typically, lower dimensional invariant tori, which are of saddle type, and that were called *whiskered* tori by Arnold because they have associated unstable and stable invariant manifolds.

Nekhoroshev theorem, first stated in 1977, establishes *Effective stability* for all the trajectories of a steep nearly-integrable system: For every initial condition $(\varphi(0), I(0))$ one has an estimate of the type

$$|I(t) - I(0)| \leq r_0\,\varepsilon^b \qquad \text{for } |t| \leq T_0 \exp\left\{(\varepsilon_0/\varepsilon)^a\right\}.$$

The constants $a, b > 0$ are called *stability exponents.*
If h is quasiconvex, that is, for any $I \in G$ and $v \in \mathbb{R}^n$,

$$Dh(I)v = 0 \text{ and } v \neq 0 \quad \Longrightarrow v^\top D^2 h(I) v \neq 0.$$

$a = b = \frac{1}{2n}$.

$$H^*(q,p,\varphi,I,s,A) = \frac{1}{2}\left(p^2 + I^2\right) + A + \varepsilon(\cos q - 1)\left(1 + \mu(\sin\varphi + \cos s)\right),$$

Since $h(p,I,A) = \frac{1}{2}\left(p^2 + I^2\right) + A$ satisfies $\begin{vmatrix} D^2h & Dh \\ Dh^\top & 0 \end{vmatrix} = -1 < 0$, one can check that h is quasiperiodic, and a priori

$$|(p,I,A)(t) - (p,I,A)(0)| \le r_0\,\varepsilon^{1/6} \qquad \text{for } |t| \le T_0 \exp\left\{(\varepsilon_0/\varepsilon)^{1/6}\right\}.$$

A refinement [Pöschel93, D-Gutiérrez96] for orbits close to the *single resonance* $p = 0$, using resonant normal forms, gives

$$|I(t) - I(0)| \le r_0\,\varepsilon^{1/4} \qquad \text{for } |t| \le T_0 \exp\left\{(\varepsilon_0/\varepsilon)^{1/4}\right\}.$$

For a *nearly-integrable* Hamiltonian with $n+1$ degrees of freedom

$$H(\varphi, I) = h(I) + \varepsilon f(\varphi, I), \qquad (\varphi, I) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$$

Select $I^* = 0$, and assume that the associated frequency vector $\lambda^* = \partial_I h(0) \in \mathbb{R}^{n+1}$ has a *single resonance*: $\langle k^*, \lambda^* \rangle = 0$ for some $0 \neq k^* \in \mathbb{Z}^{n+1}$ and $\langle k, \lambda^* \rangle \neq 0$ for any $k \in \mathbb{Z}^{n+1}$ not co-linear to k^*.
By a classical algebraic result, we can assume λ^* of the form

$$\lambda^* = (0, \omega^*) \,,$$

where $\omega^* \in \mathbb{R}^n$ is non-resonant. (In fact, we shall assume a Diophantine condition on ω^* to apply later on KAM theorem).
The unperturbed Hamiltonian can be written (up to a constant) as:

$$h(I) = \langle \lambda^*, I \rangle + \frac{1}{2} \langle QI, I \rangle + \mathrm{O}_3(I).$$

Replace $\varphi \to (q, \varphi)$ and $I \to (p, I)$, and thus split $(\varphi, I) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ as $(q, p, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}^n$, and the matrix $Q = \partial_I^2 h(0)$ as

$$\partial_{p,I}^2 h(0) = \begin{pmatrix} \beta^2 & \lambda^\top \\ \lambda & Q \end{pmatrix},$$

where we have put $\beta^2 > 0$ in order to fix ideas, $\lambda \in \mathbb{R}^n$ is a *shift* vector, and the new matrix Q is $n \times n$. We will assume $\beta = 1$; this can be achieved replacing p, I by p/β, I/β (changing in this way the time scale by a factor β), and rewriting ω^*/β, λ/β^2, Q/β^2 as ω^*, λ, Q respectively, and redefining also the function f.

Then, we can write our Hamiltonian in the form

$$H(q,p,\varphi,I) = h(p,I) + \varepsilon f(q,p,\varphi,I),$$
$$h(p,I) = \langle \omega^*, I\rangle + \frac{p^2}{2} + \langle \lambda, I\rangle\, p + \frac{1}{2}\langle QI, I\rangle + \mathrm{O}_3(p,I).$$

We now perform *one* step of resonant normal form procedure: following the Lie method, we seek for functions $S(q,\varphi)$ and $R(q,p,\varphi,I) = \mathrm{O}(p,I)$ such that

$$\{S,h\} + V + R = f, \tag{2}$$

where $V(q)$ is the periodic function obtained by averaging $f(q,0,\varphi,0)$ with respect to the angles φ:

$$V(q) = \overline{f(q,0,\cdot,0)} = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(q,0,\varphi,0)\mathrm{d}\varphi, \qquad q \in \mathbb{T}.$$

The construction of S and R is easily carried out: one first solves the equation

$$\langle \omega^*, \partial_\varphi S \rangle + V = f(\cdot, 0, \cdot, 0)$$

with the help of standard small divisors estimates, and then one takes R simply by fitting equation (2). The time-1 symplectic flow Φ of the generating Hamiltonian εS leads to

$$H \circ \Phi = H + \{H, \varepsilon S\} + \mathrm{O}\left(\varepsilon^2\right) = h + \varepsilon(V + R) + \mathrm{O}\left(\varepsilon^2\right) = H_0 + H_1,$$

with

$$\begin{aligned} H_0(q, p, I; \varepsilon) &= \langle \omega^*, I \rangle + \frac{p^2}{2} + \varepsilon V(q) + \langle \lambda, I \rangle\, p + \frac{1}{2} \langle QI, I \rangle\,, \\ H_1(q, p, \varphi, I; \varepsilon) &= \varepsilon R(q, p, \varphi, I) + \mathrm{O}_3(p, I) + \mathrm{O}\left(\varepsilon^2\right). \end{aligned}$$

Note: $\omega^* = \lambda = 0$, $V(q) = \cos q - 1$, $H_1 = \mathrm{O}(\varepsilon\mu)$ in the Arnold example.

This expression generalizes Arnold's example.
Concerning V, except for degenerate cases, the function $V(q)$ will have a unique and nondegenerate maximum q_0; we denote $\alpha^2 = -V''(q_0) > 0$.
Then, for $\varepsilon > 0$, the 1-degree-of-freedom Hamiltonian

$$P(q, p; \varepsilon) = \frac{p^2}{2} + \varepsilon V(q),$$

has a saddle point in $(q_0, 0)$, with (homoclinic) separatrices. The case $\varepsilon < 0$ is analogous, provided one considers a minimum instead of a maximum. Then, the Hamiltonian H_0 has whiskered tori with coincident whiskers associated to this saddle point.
Note that H_0 constitutes a Hamiltonian situated between the unperturbed Hamiltonian h and the perturbed one H, which possesses hyperbolic invariant tori but their whiskers still coincide.
Note also that, *in general*, H_0 is not an uncoupled Hamiltonian because of the *coupling term* $\langle \lambda, I \rangle \, p$.

The Lyapunov exponents of the saddle point of the "pendulum" P are $\pm\sqrt{\varepsilon}\alpha$, which tend to zero for $\varepsilon \to 0^+$.
To have fixed Lyapunov exponents, we can replace p, I by $\sqrt{\varepsilon}p$, $\sqrt{\varepsilon}I$.
The new system is still Hamiltonian if we divide the Hamiltonian by ε (making in this way a change of time scale by a factor $\sqrt{\varepsilon}$):

$$H_0 = \langle \omega, I \rangle + \frac{p^2}{2} + V(q) + \langle \lambda, I \rangle p + \frac{1}{2} \langle QI, I \rangle, \tag{3}$$

$$H_1 = R\left(x, \sqrt{\varepsilon}y, \varphi, \sqrt{\varepsilon}I\right) + \frac{1}{\varepsilon} \mathrm{O}_3\left(\sqrt{\varepsilon}y, \sqrt{\varepsilon}I\right) + \mathrm{O}(\varepsilon) = \mathrm{O}(\mu), \tag{4}$$

where

$$\omega = \frac{\omega^*}{\sqrt{\varepsilon}}, \qquad \mu = \sqrt{\varepsilon}.$$

For $\varepsilon \to 0^+$, the study of the Hamiltonian (3–4) is a singular perturbation problem, due to the *fast frequencies* $\omega = \omega^*/\sqrt{\varepsilon}$ in the unperturbed Hamiltonian H_0. We are thus confronted with a *singular* system, often referred to as *weakly hyperbolic*, and also called *a-priori stable* [Chierchia-Gallavotti94]. In fact, this case can be referred to as *totally singular*, because *all* the frequencies are fast.

The singular problem can be avoided if one considers independent parameters, namely a *fixed* $\varepsilon > 0$ (that is, a *fixed* ω in (3)) and $\mu \to 0$. In such a case, the system (3–4) has the property that the hyperbolicity and the homoclinic orbits are present in the unperturbed Hamiltonian ($\mu = 0$), and are simply perturbed for $|\mu|$ small. In this case, we are confronted with a *regular* or *strongly hyperbolic* system, or also *a-priori unstable*.

This strategy of keeping $\varepsilon > 0$ fixed and letting $\mu \to 0$ was introduced by Poincaré in 1889 and followed in Arnold's example to avoid dealing with a singular perturbation problem.

Unfortunately, the *exponentially small splitting of separatrices* predicted by a direct application of the Poincaré-Arnold-Melnikov (PMA) method

$$\text{Splitting distance} = \varepsilon \text{ PMA prediction} + \mathrm{O}(\varepsilon\mu)$$

when the PMA prediction $= \mathrm{O}\left(\mathrm{e}^{-c/\varepsilon^a}\right)$ could then be justified only for μ exponentially small in ε.

$$H(q,p,\varphi,I,s) = \frac{1}{2}p^2 + \varepsilon(\cos q - 1) + \frac{1}{2}I^2 + \varepsilon\mu f(q) g(\varphi,s)$$

$$f(q) = \cos q - 1, \qquad g(\varphi,s) = \sin\varphi + \cos s,$$

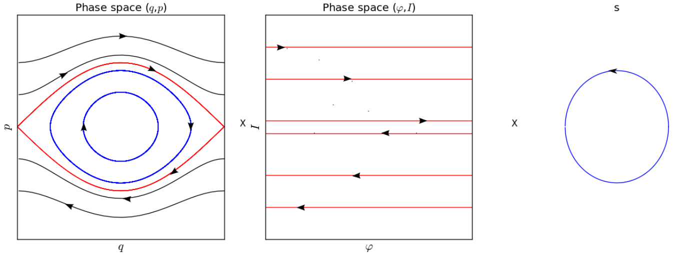


Figure: Phase Space - Unperturbed problem for $\varepsilon = 0$

- Invariant tori (2D)

$$\widetilde{\mathcal{T}}_I = \{(0, 0, I, \varphi, s) : (\varphi, s) \in \mathbb{T}^2\}$$

- Invariant manifolds (3D):

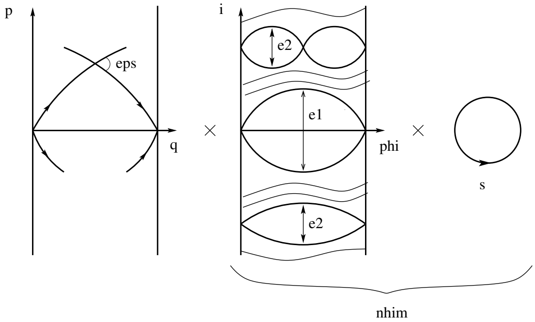
$$W^s\widetilde{\mathcal{T}}_I = W^u\widetilde{\mathcal{T}}_I = \{(q_0(\sqrt{\varepsilon}\tau), \sqrt{\varepsilon}p_0(\sqrt{\varepsilon}\tau), I, \varphi, s) : \tau \in \mathbb{R}, (\varphi, s) \in \mathbb{T}^2\}$$

where

$$q_0(t) = 4 \arctan \mathrm{e}^{\pm t}, \quad p_0(t) = 2/\cosh t.$$

is the separatrix for positive p of the standard pendulum

$$P(q, p) = p^2/2 + \cos q - 1.$$



- By the special form of the perturbation, $\widetilde{\mathcal{T}}_I$ persist to $\widetilde{\mathcal{T}}_I^{\varepsilon} = \widetilde{\mathcal{T}}_I$
- $W^s\widetilde{\mathcal{T}}_I^{\varepsilon}$ and $W^u\widetilde{\mathcal{T}}_I^{\varepsilon}$ are ε-close to the unperturbed ones.
- Using Poincaré-Melnikov theory, $W^s\widetilde{\mathcal{T}}_I^{\varepsilon} \pitchfork W^u\widetilde{\mathcal{T}}_I^{\varepsilon}$ with an angle of size $\mathrm{e}^{-\pi\sqrt{\varepsilon}/2}$.
- Therefore $W^s\widetilde{\mathcal{T}}_{I_i}^{\varepsilon} \pitchfork W^u\widetilde{\mathcal{T}}_{I_{i+1}}^{\varepsilon}$ for $|I_i - I_{i+1}| \le \mathrm{e}^{-\pi\sqrt{\varepsilon}/2}$ and a shadowing (*transition chain mechanism*) gives the diffusion path.

Minor 4 pages paper in *Dokl. Akad. Nauk SSSR*. "The details of the proof must be formidable, although the idea of the proof is clearly outlined." (J. Moser in the *MathSciNet* review)

Fixable The perturbation maintains fixed *all* the invariant tori $\mathcal{T}_I$. In general, there appear gaps around resonant tori (rational I) which prevent $W^s\widetilde{\mathcal{T}}^{\varepsilon}_{I_i} \pitchfork W^u\widetilde{\mathcal{T}}^{\varepsilon}_{I_{i+1}}$ because $\widetilde{\mathcal{T}}^{\varepsilon}_{I_i}$ and $\widetilde{\mathcal{T}}^{\varepsilon}_{I_{i+1}}$ are too far. The Scattering map can fix it.

Major The exponentially small size of the splitting $e^{-\pi\sqrt{\varepsilon}/2}$ computed from a direct application of the PMA method is much less than the Nekhoroshev estimates $e^{-\pi\varepsilon^{1/4}/2}$.

Major Arnold example only shows global instability along a single resonance, where the associated normal form is integrable, but does not deal with multiple resonances, where the normal form is not integrable.

Exponentially small splitting of separatrices

The exponentially small splitting of separatrices was already found by Poincaré in 1890, and first addressed in 1984 by Neishtadt with upper bounds using normal forms and by Lazutkin with asimptotic estimates using complex parameterizations of the stable and unstable invariant manifolds.

Proofs of its asymptotic behavior for the rapidly forced pendulum or other rapidly oscillating periodic perturbations in [D-Seara92,Gelfreich93,Fontich93-95,Sauzin95,Treschev97,D-Seara97,Gelfreich97,Baldomá-Fontich04-06,Guardia-Olivé-Seara10,Baldomá-Fontich-Guardia-Seara12].

For maps, upper exponentially small estimates in [Fontich-Simó90] and asymptotic estimates in [D-Ramírez-Ros98-99,Simó-Vieiro09,Martín-Sauzin-Seara11].

Exponentially small splitting of separatrices

In the rapidly quasiperiodically forced pendulum, the rôle of the arithmetic properties was detected in [Sim94], and established in [D-Gelfreich-Seara-Jorba97].

For n-dimensional whiskered tori of a Hamiltonian with $n+1$ degrees of freedom, the splitting potential and Melnikov potential were introduced [Eliasson94,D-Gutiérrez00], sharp exponentially small upper bounds were given in [D-Gutiérrez-Seara04], and asymptotic estimates in [Lochak-Marco-Sauzin03,D-Gutiérrez04,D-GonchenkoGutiérrez14-16].

The multidimensional *separatrix map* introduced by Treschev in 2002 requires more study.

We consider a 2π-periodic in time perturbation of a pendulum and a rotor described by the non-autonomous Hamiltonian,

$$\begin{aligned} H_\varepsilon(p,q,I,\varphi,t) &= H_0(p,q,I) + \varepsilon h(p,q,I,\varphi,t;\varepsilon) \\ &= P_\pm(p,q) + \tfrac{1}{2}I^2 + \varepsilon h(p,q,I,\varphi,t;\varepsilon) \end{aligned} \tag{5}$$

where $(p,q,I,\varphi,t) \in (\mathbb{R}\times\mathbb{T})^2 \times \mathbb{T}$ and

$$P_\pm(p,q) = \pm\left(\frac{1}{2}p^2 + V(q)\right) \tag{6}$$

and $V(q)$ is a 2π-periodic function. We will refer to $P_\pm(p,q)$ as the *pendulum*.

Note. This model just comes from a normal form around a single resonance of a nearly integrable Hamiltonian. The perturbation is arbitrary.

Theorem (D-Llave-Seara06)

Consider the Hamiltonian (5) where V and h are uniformly $\mathcal{C}^{r+2}$ for $r \geq r_0$, sufficiently large. Assume also that

H1 *The potential $V : \mathbb{T} \to \mathbb{R}$ has a unique global maximum at $q = 0$ which is non-degenerate. Denote by $(q_0(t), p_0(t))$ an orbit of the pendulum $P_\pm(p, q)$ homoclinic to $(0, 0)$.*

H2 *The Melnikov potential, associated to h (and to the homoclinic orbit (p_0, q_0)):*

$$\begin{aligned} \mathcal{L}(I, \varphi, s) \;=\; -\int_{-\infty}^{+\infty} & (h(p_0(\sigma), q_0(\sigma), I, \varphi + I\sigma, s + \sigma; 0) \\ & -h(0, 0, I, \varphi + I\sigma, s + \sigma; 0))d\sigma \end{aligned} \tag{7}$$

satisfies concrete non-degeneracy conditions.

H3 *The perturbation term h satisfies concrete non-degeneracy conditions.*

Then, there is $\varepsilon^* > 0$ such that for $0 < \varepsilon < \varepsilon^*$, and for any interval $[I_-^*, I_+^*]$, there exists a trajectory $\widetilde{x}(t)$ of the system (5) such that for some $T > 0$,

$$I(\widetilde{x}(0)) \leq I_-^*; \qquad I(\widetilde{x}(T)) \geq I_+^*.$$

Remark Arbitrary excursions in the I variable can also be realized.

Hypotheses **H1**, **H2** and **H3** are $\mathcal{C}^2$ generic, so, the following short version of the Theorem also holds:

Theorem (D-Huguet09)

Consider the Hamiltonian (5) and assume that V *and* h *are* $\mathcal{C}r+2$ *functions which are* $\mathcal{C}^2$ *generic, with* $r > r_0$*, large enough. Then there is* $\varepsilon^* > 0$ *such that for* $0 < |\varepsilon| < \varepsilon^*$ *and for any interval* $[I_-^*, I_+^*]$*, there exists a trajectory* $\widetilde{x}(t)$ *of the system with Hamiltonian (5) such that for some* $T > 0$

$$I(\widetilde{x}(0)) \leq I_-^*; \qquad I(\widetilde{x}(T)) \geq I_+^*.$$

Remark A (non optimal) value of r_0 which follows from our argument is $r_0 = 242$.

Consider a periodic in time perturbation of n pendula and a d-dimensional rotor described by the non-autonomous Hamiltonian,

$$H(p, q, I, \varphi, t, \varepsilon) = P(p, q) + h(I) + \varepsilon Q(p, q, I, \varphi, t, \varepsilon), \tag{8}$$

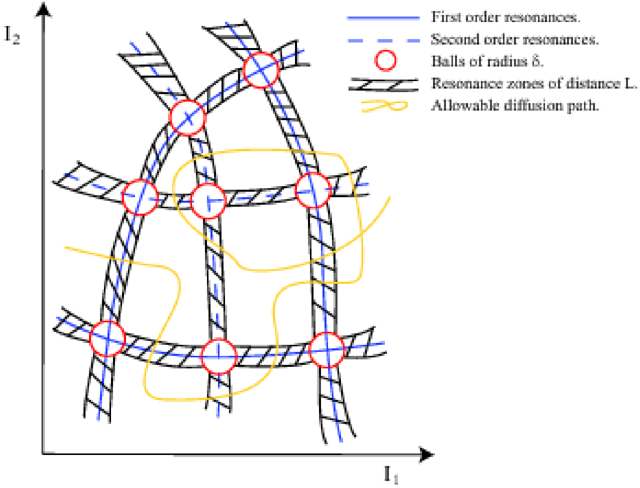
with $P(p, q) = \sum_{j=1}^{n} P_j(p_j, q_j)$, $P_j(p_j, q_j) = \pm\left(\frac{1}{2}p_j^2 + V_j(q_j)\right)$, where $I \in \mathcal{I} \subset \mathbb{R}^d, \varphi \in \mathbb{T}^d$, $\mathcal{I}$ an open set, $p, q \in \mathbb{R}^n$, $t \in \mathbb{T}^1$, and $P_j(p_j, q_j)$ is a *pendulum* for the saddle variables p_j, q_j. For $\varepsilon = 0$, the d-dimensional action I remains constant. Under similar hypotheses as for $n = d = 1$,

Theorem (D-Llave-Seara12)

For every $\delta > 0$, there exists $\varepsilon_0 > 0$, such that for every $0 < |\varepsilon| < \varepsilon_0$, given $I_\pm \in \mathcal{I}$,there exists a solution $\tilde{x}(t)$ of (8) and $T > 0$, such that

$$|I(\tilde{x}(0)) - I_-| \leq C\delta \quad \textit{and} \quad |I(\tilde{x}(T)) - I_+| \leq C\delta \tag{9}$$

- One can forget about δ and prescribe arbitrary paths on a set $\mathcal{I}^*$. This set $\mathcal{I}^*$ is described precisely in the course of the proof, and is determined by the non-degeneracy assumptions. The main idea is that $\mathcal{I}^*$ is obtained from the domain of definition, just eliminating some sets of codimension 2, like double resonances, from the open set where the intersection of stable and unstable manifolds of a normally hyperbolic invariant manifold is transversal.
- Codimension 2 objects do not separate the regions and can be contoured so that they do not obstruct the change along the paths. It seems that such contouring trajectories close to double resonances are inferred from some movies related to numerical experiments in (Gelfreich-Simó-Vieiro 13)



I_2
I_1
First order resonances.
Second order resonances.
Balls of radius δ.
Resonance zones of distance L.
Allowable diffusion path.

Other contributions

This problem of instability, also called Arnold diffusion, was posed first by Arnold in 1964, and there have been some other contributions, using geometrical or variational methods:
[Lochak92], [Chierchia-Gallavotti94-98], [Bessi-Chierchia-Valdinoci01] [Berti-Biasco-Bolle03], [Marco-Sauzin03], [Mather04], [Cheng-Yan04], [Gidea-Llave06], [Piftankin-Treschev07], [Kaloshin-Levi08], [ChengY09], [Bernard-Kaloshin-Zhang11], [Zhang11], [Mather12], [Treschev12], [Gelfreich-Simó-Vieiro13], [GelfreichT14], [Gidea-Llave-Seara14], [Kaloshin-Zhang15], [Lazzarini-Marco-SauzinS15], [Davletshin-Treschev16], [Marco16], [Gidea-Marco17], [Cheng17].

The main idea of the proof is to use the two (or more) dynamics on $\widetilde{\Lambda}$.

- Find a big invariant saddle object: a NHIM (normally hyperbolic invariant manifold: a global version of a center manifold) $\widetilde{\Lambda}$ with transverse associated stable and unstable manifolds along some homoclinic manifold Γ: $\mathcal{W}^u(\widetilde{\Lambda}) \pitchfork_\Gamma \mathcal{W}^s(\widetilde{\Lambda})$.
- Compute the invariant objects (typically tori $\mathcal{T}$) which may prevent instability for the inner dynamics of the NHIM.
- Compute an scattering map $S = S^\Gamma : H_- \subset \widetilde{\Lambda} \to H_+ \subset \widetilde{\Lambda}$ on the NHIM associated to Γ and consider it as an outer dynamics on the NHIM (a second dynamics on Γ).
- Check that $S(\mathcal{T}_{I_i}) \pitchfork \mathcal{T}_{I_{i+1}}$ for a sequence of tori $\{\mathcal{T}_{I_i}\}_{i=1}^N$ with $|I_N - I_1| = \mathcal{O}(1)$, and construct a transition chain of whiskered tori, i.e. $\mathcal{W}^u(\mathcal{T}_{I_i}) \pitchfork \mathcal{W}^s(\mathcal{T}_{I_{i+1}})$.
- Standard shadowing methods provide an orbit that follows closely the transition chain.

Consider a pendulum and a rotor plus a time periodic perturbation depending on two harmonics in the variables (φ, s):

$$H_\varepsilon(p, q, I, \varphi, t) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \varepsilon h(q, \varphi, s) \qquad (10)$$

$$\begin{gathered} h(q, \varphi, s) = f(q) g(\varphi, s), \\ f(q) = \cos q, \qquad g(\varphi, s) = a_1 \cos(k_1\varphi + l_1 s) + a_2 \cos(k_2\varphi + l_2 s), \end{gathered} \qquad (11)$$

with $k_1,\ k_2,\ l_1,\ l_2 \in \mathbb{Z}$.

Theorem

Assume that $a_1 a_2 \neq 0$ *and* $\begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix} \neq 0$ *in (10)-(11). Then, for any* $I^* > 0$, *there exists* $\varepsilon^* = \varepsilon^*(I^*, a_1, a_2) > 0$ *such that for any* ε, $0 < \varepsilon < \varepsilon^*$, *there exists a trajectory* $(p(t), q(t), I(t), \varphi(t))$ *such that for some* $T > 0$

$$I(0) \leq -I^* < I^* \leq I(T).$$

We have two important dynamics associated to the system: the inner and the outer dynamics.

$$\widetilde{\Lambda} = \{(0,0,I,\varphi,s); I \in [-I^*, I^*], (\varphi,s) \in \mathbb{T}^2\}.$$

is a 3D *Normally Hyperbolic Invariant Manifold* (NHIM) with associated 4D stable $W^{\mathrm{s}}_{\varepsilon}(\widetilde{\Lambda})$ and unstable $W^{\mathrm{u}}_{\varepsilon}(\widetilde{\Lambda})$ invariant manifolds.

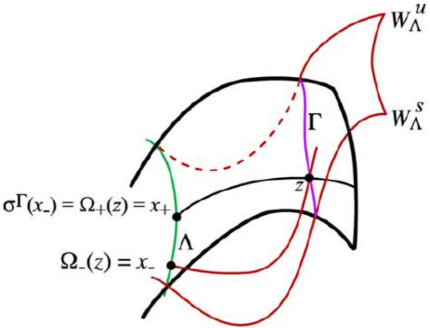
- The *inner dynamics* is the dynamics restricted to $\widetilde{\Lambda}$. (Inner map)
- The *outer dynamics* is the dynamics restricted to its invariant manifolds. (Scattering map)

Remark: for simplicity, in our case $\widetilde{\Lambda} = \widetilde{\Lambda}_{\varepsilon}$.

Let $\widetilde{\Lambda}$ be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold Γ. A scattering map is a map S defined by $S(\tilde{x}_-) = \tilde{x}_+$ if there exists $\tilde{z} \in \Gamma$ satisfying

$$|\phi_t^\varepsilon(\tilde{z}) - \phi_t^\varepsilon(\tilde{x}_\mp)| \longrightarrow 0 \text{ as } t \longrightarrow \mp\infty$$

that is, $W_\varepsilon^u(\tilde{x}_-)$ intersects transversally $W_\varepsilon^s(\tilde{x}_+)$ in $\tilde{z}$.



S is symplectic and exact (Delshams -de la Llave - Seara 2008) and takes the form:

$$S_\varepsilon(I,\varphi,s) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I,\theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I,\theta) + \mathcal{O}(\varepsilon^2), s \right),$$

where $\theta = \varphi - Is$ and $\mathcal{L}^*(I,\theta)$ is the Reduced Poincaré function, or more simply in the variables (I,θ):

$$\mathcal{S}_\varepsilon(I,\theta) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I,\theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I,\theta) + \mathcal{O}(\varepsilon^2) \right),$$

- The variable s remains fixed under S_ε: it plays the role of a parameter
- Up to first order in ε, $\mathcal{S}_\varepsilon$ is the $-\varepsilon$-time flow of the Hamiltonian $\mathcal{L}^*(I,\theta)$
- The scattering map jumps $\mathcal{O}(\varepsilon)$ distances along the level curves of $\mathcal{L}^*(I,\theta)$

To get a scattering map we search for homoclinic orbits to $\tilde{\Lambda}_\varepsilon$

Proposition

Given $(I,\varphi,s) \in [-I^*,I^*] \times \mathbb{T}^2$, assume that the real function

$$\tau \in \mathbb{R} \longmapsto \mathcal{L}(I,\varphi - I\tau, s-\tau) \in \mathbb{R}$$

has a non degenerate critical point $\tau^* = \tau(I,\varphi,s)$, where $\mathcal{L}(I,\varphi,s) =$

$$\int_{-\infty}^{+\infty} h(p_0(\sigma),q_0(\sigma),I,\varphi+I\sigma,s+\sigma;0) - h(0,0,I,\varphi+I\sigma,s+\sigma;0)d\sigma.$$

Then, for $0 < |\varepsilon|$ small enough, there exists a transversal homoclinic point $\tilde{z}$ to $\widetilde{\Lambda}_\varepsilon$, which is ε-close to the point $\tilde{z}^*(I,\varphi,s) = (p_0(\tau^*),q_0(\tau^*),I,\varphi,s) \in W^0(\widetilde{\Lambda})$:

$$\tilde{z} = \tilde{z}(I,\varphi,s) = (p_0(\tau^*)+O(\varepsilon),q_0(\tau^*)+O(\varepsilon),I,\varphi,s) \in W^u(\widetilde{\Lambda}_\varepsilon) \pitchfork W^s(\widetilde{\Lambda}_\varepsilon).$$

In our model the perturbation is

$$h(p,q,I,\varphi,s) = \cos q\,(a_0\cos(k_1\varphi + l_1 s) + a_1\cos(k_2\varphi + l_2 s))$$

and the Melnikov potential becomes

$$\mathcal{L}(I,\varphi,s) = A_0(I)\cos(k_1\varphi + l_1 s) + A_1(I)\cos(k_2\varphi + l_2 s),$$

where $A_0(I) = \dfrac{2\,\pi\,(k_1 I + l_1)\,a_0}{\sinh(\frac{(k_1 I + l_1)\,\pi}{2})}$ and $A_1 = \dfrac{2\,(k_2 I + l_2)\pi\,a_1}{\sinh(\frac{(k_2 I + l_2)\pi}{2})}$.

Definition

The Reduced Poincaré function is

$$\mathcal{L}^*(I,\theta) = \mathcal{L}(I,\varphi - I\,\tau^*(I,\varphi,s), s - \tau^*(I,\varphi,s)),$$

where $\theta = \varphi - I\,s$.

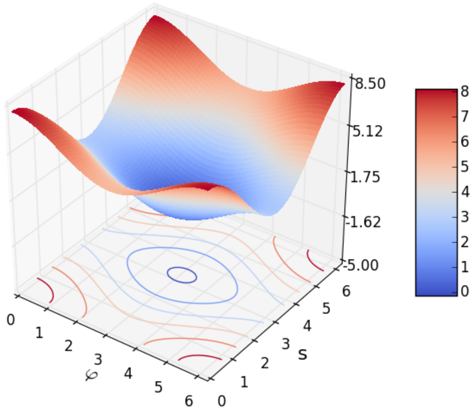


Figure: The Melnikov Potential, $\mu = a_0/a_1 = 0.6$, $I = 1$, $k_1 = l_2 = 1$ and $k_2 = l_1 = 0$.

We look for τ^* such that $\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\,\tau^*, s - \tau^*) = 0$.

Different view-points for $\tau^* = \tau^*(I, \varphi, s)$

- Look for critical points of $\mathcal{L}$ on the straight line $R(I, \varphi, s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$.
- Look for intersections between $R(I, \varphi, s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$ and a crest which is a curve of equation

$$\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\tau, s - \tau)|_{\tau=0} = 0.$$

Definition - Crests (Delshams-Huguet 2011)

For each I, we call *crest* $\mathcal{C}(I)$ the set of curves in the variables (φ, s) of equation

$$I\frac{\partial\mathcal{L}}{\partial\varphi}(I,\varphi,s)+\frac{\partial\mathcal{L}}{\partial s}(I,\varphi,s)=0. \tag{12}$$

which in our case can be rewritten as

$$\mu\alpha(I)\sin\varphi+\sin s=0, \qquad \text{with } \alpha(I)=\frac{\sinh(\frac{\pi}{2})\,I^2}{\sinh(\frac{\pi I}{2})}, \quad \mu=\frac{a_{10}}{a_{01}}. \tag{13}$$

- For any I, the critical points of the Melnikov potential $\mathcal{L}(I,\cdot,\cdot)$ ($(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π): one maximum, one minimum point and two saddle points) always belong to the crest $\mathcal{C}(I)$.
- $\mathcal{L}^*(I,\theta)$ is nothing else but $\mathcal{L}$ evaluated on the crest $\mathcal{C}(I)$.
- $\theta=\varphi-Is$ is constant on the straight line $R(I,\varphi,s)$

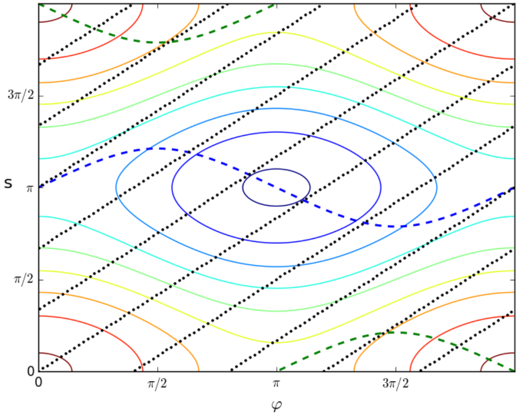


Figure: Level curves of $\mathcal{L}$ for $\mu = a_0/a_1 = 0.5$, $I = 1.2$, $k_1 = l_2 = 1$ and $k_2 = l_1 = 0$.

Understanding the behavior of the crests

$\Downarrow$

Understanding the behavior of the Reduced Poincaré function

$\Downarrow$

Understanding the Scattering map

We only need to study two cases:

- The first (easier) case [D-Schaefer 17]

$$h(q, \varphi, s) = \cos q \,(a_0 \cos \varphi + a_1 \cos s)$$

- The second case [D-Schaefer 17]

$$h(q, \varphi, s) = \cos q \,(a_0 \cos \varphi + a_1 \cos(\varphi - s))$$

Each case has its own characteristics and together are enough to understand the general case.
We present just some highlights about each case.

Definition: Highways

Highways are the level curves of $\mathcal{L}^*$ such that

$$\mathcal{L}^*(I, \theta) = \frac{2\pi a_0}{\sinh(\pi/2)}.$$

- The highways are "vertical" in the variables (φ, s)
- We always have a pair of highways. One goes up, the other goes down (this depends on the sign of $\mu = a_0/a_1$)
- The highways give rise to fast diffusing pseudo-orbits

Plot of highways

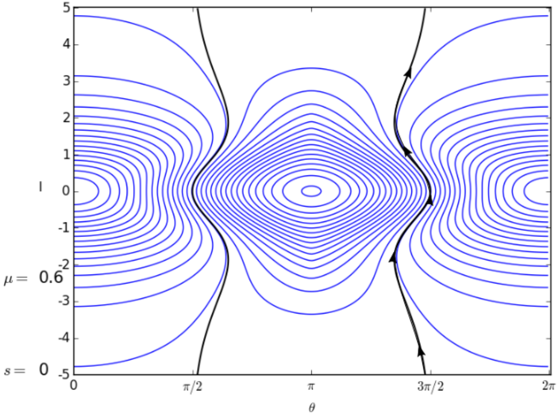
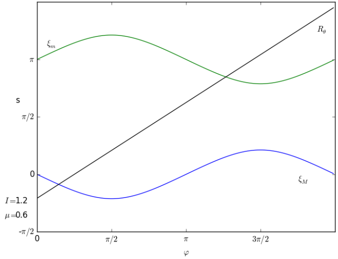


Figure: The scattering map jumps $\mathcal{O}(\varepsilon)$ distances along the level curves of $\mathcal{L}^*(I, \theta)$

- For $|\mu\alpha(I)| < 1$, there are two crests $\mathcal{C}_{\mathrm{M,m}}(I)$ parameterized by:

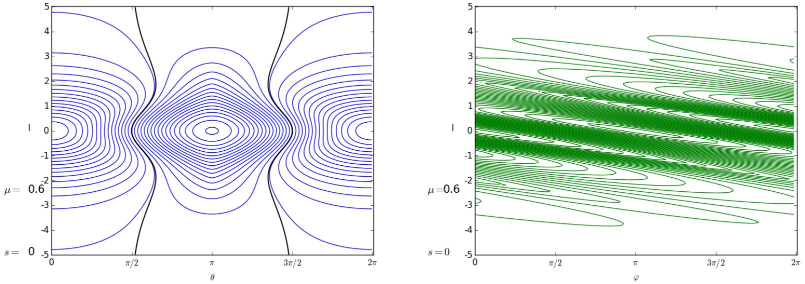
$$
\begin{aligned}
s = \xi_M(I,\varphi) &= -\arcsin(\mu\alpha(I)\sin\varphi) \mod 2\pi \\
\xi_m(I,\varphi) &= \arcsin(\mu\alpha(I)\sin\varphi) + \pi \mod 2\pi
\end{aligned}
\tag{14}
$$



They are "horizontal" crests

$k_1 = l_2 = 1$ and $k_2 = l_1 = 0$ $\quad 0 < |\mu| < 0.625$

- For each I, the line $R(I, \varphi, s)$ and the crest $\mathcal{C}_{\mathrm{M,m}}(I)$ have only one intersection point.
- The scattering map S_{M} associated to the intersections between $\mathcal{C}_{\mathrm{M}}(I)$ and $R(I, \varphi, s)$ is well defined for any $\varphi \in \mathbb{T}$. Analogously for S_{m}, changing M to m. In the variables $(I, \theta = \varphi - Is)$, both scattering maps $\mathcal{S}_{\mathrm{M}}$, $\mathcal{S}_{\mathrm{m}}$ are globally well defined.

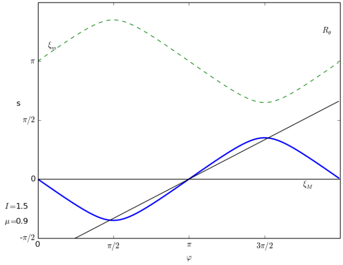


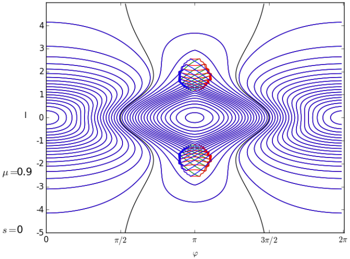
(a) Level curves of $\mathcal{L}^*_M(I, \theta)$

(b) Level curves of $\mathcal{L}^*_m(I, \theta)$

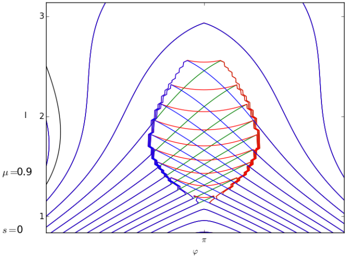
$k_1 = l_2 = 1$ and $k_2 = l_1 = 0$ $\quad 0.625 < |\mu|$

- There are tangencies between $\mathcal{C}_{\mathrm{M,m}}(I, \varphi)$ and $R(I, \varphi, s)$. For some value of (I, φ, s), there are 3 points in $R(I, \varphi, s) \cap \mathcal{C}_{\mathrm{M,m}}(I)$.
- This implies that there are 3 scattering maps associated to each crest with different domains.(Multiple Scattering maps)





(c) The three types of level curves.

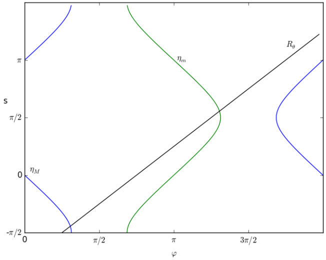


(d) Zoom where the scattering maps are different

Figure: Level curves of $\mathcal{L}^*_M(I,\theta)$, $\mathcal{L}^{*(1)}_M(I,\theta)$ and $\mathcal{L}^{*(2)}_M(I,\theta)$

- For some values of I, $|\mu\alpha(I)| > 1$, the two crests $\mathcal{C}_{\mathrm{M,m}}$ are parameterized by:

$$\begin{aligned} \varphi = \eta_M(I,s) &= -\arcsin(\mu\alpha(I)\sin s) \mod 2\pi \\ \eta_m(I,s) &= \arcsin(\mu\alpha(I)\sin s) + \pi \mod 2\pi \end{aligned} \tag{15}$$



They are "vertical" crests

For the values of I and when horizontal crests become vertical, it is not always possible to prolong in a continuous way the scattering maps, so the domain of the scattering map has to be restricted.

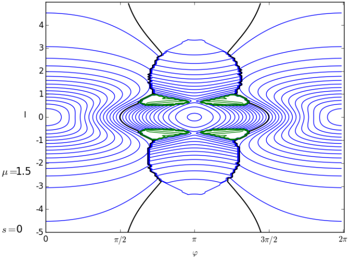


Figure: The level curves of $\mathcal{L}^*_{\mathrm{M}}(I, \theta)$, $\mu = 1.5$.

In green, the region where the scattering map S_{M} is not defined.

An example of pseudo-orbit

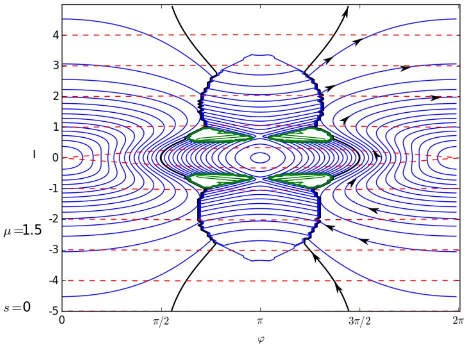


Figure: In red: Inner map, blue: Scattering map, black: Highways

An estimate of the total time of diffusion between $-I^*$ and I^*, along the highway, is

$$T_d = \frac{T_s}{\varepsilon}\left[2\log\left(\frac{C}{\varepsilon}\right) + \mathcal{O}(\varepsilon^b)\right], \text{ for } \varepsilon \to 0, \text{ where } 0 < b < 1,$$

with

$$T_s = T_s(I^*, a_{10}, a_{01}) = \int_0^{I^*} \frac{-\sinh(\pi I/2)}{\pi a_{10} I \sin\psi_h(I)}\, dI,$$

where $\psi_h = \theta - I\tau^*(I,\theta)$ is the parameterization of the highway $\mathcal{L}^*(I,\psi_h) = A_{00} + A_{01}$, and

$$C = C(I^*, a_{10}, a_{01}) = 16\,|a_{10}|\left(1 + \frac{1.465}{\sqrt{1-\mu^2 A^2}}\right)$$

where $A = \max_{I\in[0,I^*]} \alpha(I)$, with $\alpha(I) = \frac{\sinh(\frac{\pi}{2})\, I^2}{\sinh(\frac{\pi I}{2})}$ and $\mu = a_{10}/a_{01}$.

Note: This estimate quantifies the general optimal diffusion estimate $\mathcal{O}\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ of [Berti-Biasco-Bolle 2003], [Cresson-Guillet 2003] and [Treschev 2004).

Main differences between the first and the second case

In the second case:

- There are no *Highways*.
- For any value of $\mu = a_0/a_1$ is possible to find I_h and I_v such that for $I = I_\mathrm{h}$ the crests are horizontal and for $I = I_\mathrm{v}$ the crests are vertical.
- For any value of μ there exists I such that the crests and $R(I, \varphi, s)$ are tangent.

The choice of the concrete curve of the crest and therefore of $\tau^*(I,\theta)$ is very important and useful.

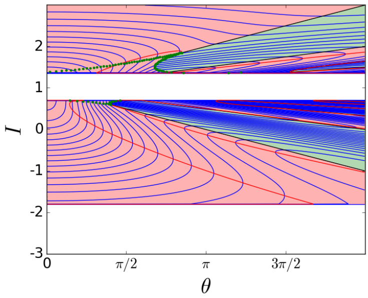


Figure: The "lower" crest

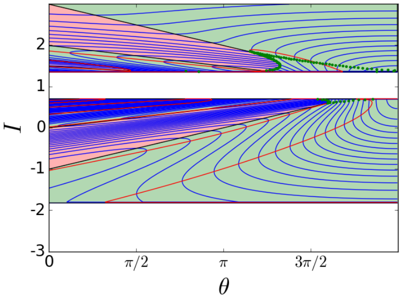
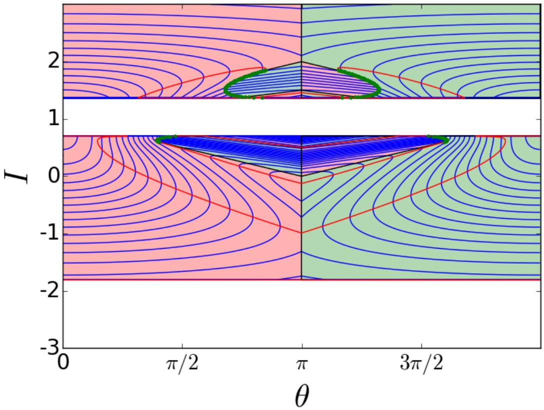


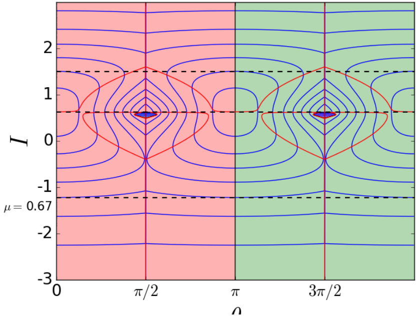
Figure: The "upper" crest

Green zones: I increases under the scattering map.
Red zones: I decreases under the scattering map.

Figure: Lower $|\tau^*|$ between "lower" and "upper" crest



In this picture we show a combination of 6 scattering maps.



$$H(I_1, I_2, \varphi_1, \varphi_2, p, q, t, \varepsilon) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + h(I_1, I_2) + \varepsilon \cos q\, g(\varphi_1, \varphi_2, t),$$

where

$$h(I_1, I_2) = \Omega_1 \frac{I_1^2}{2} + \Omega_2 \frac{I_2^2}{2}$$

and

$$g(\varphi_1, \varphi_2, t) = a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + a_3 \cos(\varphi_1 + \varphi_2 - t).$$

Under general conditions for $a_1, a_2, a_3, \Omega_1, \Omega_2$, global instability was established in [D-Llave-Seara 2016]

In this case, the Melnikov potential is

$$\mathcal{L}(I, \varphi - \omega\tau) = \sum_{i=1}^{3} A_i \cos(\varphi_i - \omega_i\tau),$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, $\omega = (\omega_1, \omega_2, \omega_3), \varphi_3 = \varphi_1 + \varphi_2 - s$,

$$A_i = \frac{2\,\pi\omega_i}{\sinh(\frac{\pi\,\omega_i}{2})} a_i,$$

and

$$\omega_1 = \Omega_1 I_1 \quad \omega_2 = \Omega_2 I_2 \quad \omega_3 = \omega_1 + \omega_2 - 1.$$

Remark: The reduced Poincaré function $\mathcal{L}^*(I, \theta)$ can be defined but the associated Hamiltonian vector field is no longer integrable

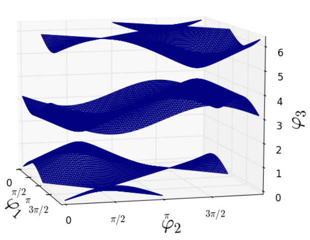


Figure: Horizontal crests: $\mu_1 = \mu_2 = 0.48$,$\omega_1 = \omega_2 = 1.219$.

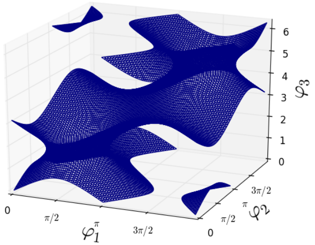


Figure: Crests with holes : $\mu_1 = 0.7, \mu_2 = 0.6$,$\omega_1 = \omega_2 = 1.219$.

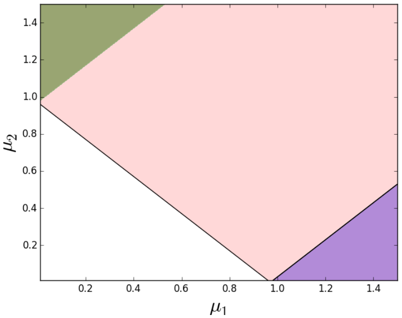


Figure: $\omega_1 = \omega_2 = 1.219$

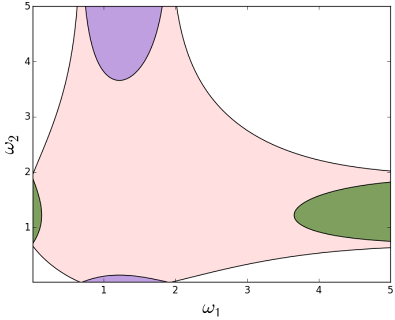


Figure: $\mu_1 = \mu_2 = 1.2$

Pink: Surface with holes, white: horizontal surfaces $s(\varphi_1, \varphi_2)$, purple: vertical surfaces $\varphi_1(\varphi_2, s)$, green: vertical surfaces $\varphi_2(\varphi_1, s)$.

(Quasi)-periodic perturbations of geodesic flows

Theorem ([D-Llave-Seara06])

Let M be a n-dimensional manifold, g a $\mathcal{C}^r$ metric on it (r sufficiently large). Assume:

H1 *There exists a closed geodesic "Λ" such that its corresponding periodic orbit $\hat{\Lambda}$ under the geodesic flow is hyperbolic.*

H2 *There exists another geodesic "γ" such that $\hat{\gamma}$ is a transversal homoclinic orbit to $\hat{\Lambda}$.*
That is, $\hat{\gamma}$ is contained in the intersection of the stable and unstable manifolds of $\hat{\Lambda}$, $W^s_{\hat{\Lambda}}$, $W^u_{\hat{\Lambda}}$, in the unit tangent bundle.
Moreover, we assume that the intersection of the stable and unstable manifolds of $\hat{\Lambda}$ is transversal along $\hat{\gamma}$. That is,

$$T_{\gamma(t)} W^s_{\hat{\Lambda}} + T_{\gamma(t)} W^u_{\hat{\Lambda}} = T_{\gamma(t)} \mathbf{S}_1 M, \quad t \in \mathbb{R}.$$

Abundance of Hypoteses **H1**, **H2**

Hipotheses **H1**, **H2** are abundant:

- They are generic on $\mathbb{T}^2$ [Morse24], [Hedlund32], [Mather94].
- They hold on any closed surface of genus bigger or equal than 2, if $r \geq 2 + \delta$, $\delta > 0$. [Katok82]).
- They are generic in the $\mathcal{C}^2$ topology for any closed surface [Contreras-Paternain02].

(Quasi)-periodic perturbations of geodesic flows

Let $\nu \in \mathbb{R}^d$ be Diophantine, $r \in \mathbb{N}$ be sufficiently large (depending on τ, the Diophantine exponent of ν).
Let g be a $\mathcal{C}^r$ metric on a compact manifold M, verifying hypotheses **H1**, **H2**, and $U : M \times \mathbb{T}^d \to \mathbb{R}$ a generic $\mathcal{C}^r$ function.
Consider the time dependent Lagrangian

$$L(q, \dot{q}, \nu t) = \frac{1}{2} g^q(\dot{q}, \dot{q}) - U(q, \nu t), \tag{16}$$

where g^q denotes the metric in $\mathbf{T}_q M$.
Then, the Euler-Lagrange equation of L has a solution $q(t)$ whose energy

$$E(t) = \frac{1}{2} g^q(\dot{q}(t), \dot{q}(t)) + U(q(t), \nu t),$$

tends to infinity as $t \to \infty$.

(Planar) elliptic restricted three body problem (ERTBP)

- Consider the motion of a particle q with zero mass (comet) under the attraction of two particles q_1 (Sun, with mass $1-\mu$) and q_2 (Jupiter, with mass μ), called *primaries*, which move in elliptic orbits with eccentricity e_0 around their center of mass.
- The motion of q is described by a time-periodic Hamiltonian system, with 2 and 1/2 degrees of freedom, with Hamiltonian

$$H(q,p,t;e_0,\mu) = \frac{p^2}{2} - \frac{(1-\mu)}{|q-q_1(t,e_0)|} - \frac{\mu}{|q-q_2(t,e_0)|}.$$

- We consider the motion of the particle q (comet) when it moves outside of the orbit of the primaries along nearly parabolic orbits.
- Parameters: $0<\mu<1$, $e_0\geq 0$, small.

The two body problem: Sun-comet for $\mu = 0$

- When $\mu = 0$, the *Sun* is fixed at the origin: $q_1(t, e_0) = 0$
- The Sun q_1 and the comet q form the two-body problem.
- In polar coordinates: $q = (r\cos\alpha, r\sin\alpha)$, $\alpha \in \mathbb{T}$, $r \geq 0$, the Hamiltonian of the two body problem becomes

$$H_0(r, P_r, \alpha, G) = \frac{P_r^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r},$$

- H_0 is the energy and $G = P_\alpha$ is the angular momentum.
- H_0 and G are both first integrals of motion.
- If $H_0 = h < 0$, motions are elliptic with semi-major axis $a = 1/(-2h)$ and eccentricity $e = \sqrt{1 + 2hG^2}$.
- If $h = 0$ (which corresponds to $e = 1$) the motion is parabolic.
- The two-body problem is integrable.

Diffusion of the angular momentum G

In the elliptic restricted three body (ERTBP) problem we want to see that the angular momentum of the comet $G(t)$ can have *large changes* when the eccentricity $e_0 > 0$ and $\mu > 0$ are small enough:

Theorem (D-Kaloshin-Rosa-Seara12)

Given any $G_1, G_2 \gg 1$, there exist trajectories of the ERTBP whose angular momentum satisfies, for some $T > 0$:

$$G(0) < G_1 \qquad G(T) > G_2$$

Proven for $0 < \mu \ll e_0 \ll 1$ and any $1 \ll G_1, G_2 \ll 1/e_0$.
Likely (need still some work) for any $0 < e_0 < 1$ and $0 < \mu \ll 1$.
Remark Two different scattering maps are used in the construction of the diffusing trajectories.

Arnold's mechanism of diffusion in the spatial RTBP

- Model:
  - The spatial circular restricted three-body problem: an infinitesimal mass moves in space under the gravitational influence of two massive bodies (primaries) describing circular orbits, without exerting any influence on them
  - Focus on the dynamics near L_1, the libration point between the primaries – center$\times$center$\times$saddle
- Results:
  - There exist trajectories that change the out-of-plane amplitude (w.r. to the ecliptic) of an orbit near L_1 by a 'significant amount', via the Arnold mechanism of instability
    - abstract theorem – if certain conditions hold true then the existence of drift trajectories follows
    - verification of conditions – some analytical, some numerical
  - Related works [Samà,2004],[Terra,Simó,de Sousa Silva,2014]

Introduction

- Method:
  - There exists a normally hyperbolic invariant three-sphere
  - We construct orbits that alternatively follow segments of homoclinic trajectories (outer dynamics) with segments of trajectories restricted to the three-sphere (inner dynamics), thus mimicking Arnold's instability mechanism of transition tori[1]
  - However, we use only coarse information on the inner dynamics (Poincaré recurrence theorem), no detailed information on the invariant objects (KAM tori, Aubry-Mather sets, etc.)
  - We use a geometric method that allows for explicit construction of drifting trajectories under milder conditions on the dynamics (compared to variational methods)
  - This is a general strategy

[1] Our model is not a small perturbation of an integrable system

Reference Problem: 3D Circular RTBP

The Restricted Three Body Problem (RTBP) defined as

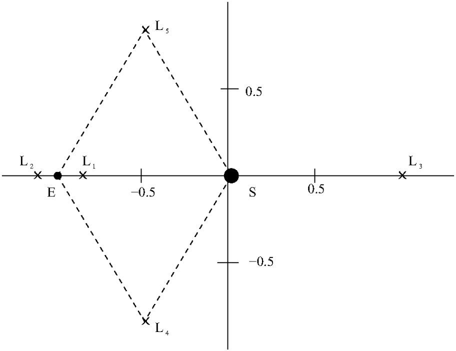
$$
\begin{aligned}
\ddot{X} - 2\dot{Y} &= \Omega_X, \\
\ddot{Y} + 2\dot{X} &= \Omega_Y, \\
\ddot{Z} &= \Omega_Z,
\end{aligned}
$$

where

$$\Omega = \frac{1}{2}(X^2 + Y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu),$$

$$
\begin{aligned}
r_1^2 &= (X-\mu)^2 + Y^2 + Z^2, \\
r_2^2 &= (X-\mu+1)^2 + Y^2 + Z^2.
\end{aligned}
$$

Libration Points



X-coordinate of L_1 is

$$X_1 = -1 + \left(\frac{\mu}{3}\right)^{1/3} - \frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3} + \mathrm{O}\,\mu.$$

In the Sun-Earth system,

Birkhoff Normal Form

On the center manifold, we obtain a two degrees of freedom Hamiltonian

$$H_c = H_N\left(0, \frac{x_2^2+y_2^2}{2}, \frac{x_3^2+y_3^2}{2}\right).$$

Define the *action-angle* coordinates

$$\begin{aligned} I_p &:= \frac{x_2^2+y_2^2}{2}, & \phi_p \\ I_v &:= \frac{x_3^2+y_3^2}{2}, & \phi_v. \end{aligned}$$

- The equations of motion are integrable

$$\dot{I}_p = 0, \qquad \dot{\phi}_p = \frac{\partial H}{\partial I_p} = \omega_p(I_p, I_v) \tag{17}$$

$$\dot{I}_v = 0, \qquad \dot{\phi}_v = \frac{\partial H}{\partial I_v} = \omega_v(I_p, I_v), \tag{18}$$

  and each solution lies on a 2-dimensional torus.
- Each torus can be identified with the actions I_p, I_v.

Family of Invariant Tori

- Let us fix the energy level to $H(0, I_p, I_v) = h$, with $H(L_1) \leq h \leq H(halo)$.
- Then we obtain a one-parameter family of invariant tori, parametrized by the vertical action I_v.

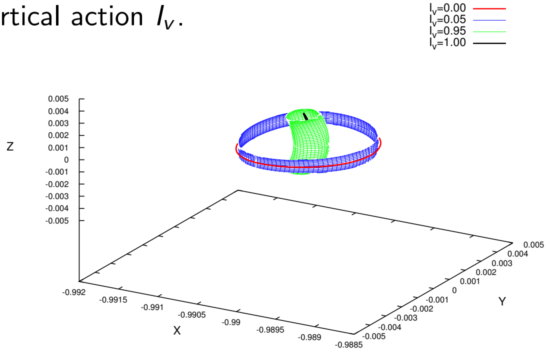


Figure: Low energy level $C = 3.00088$

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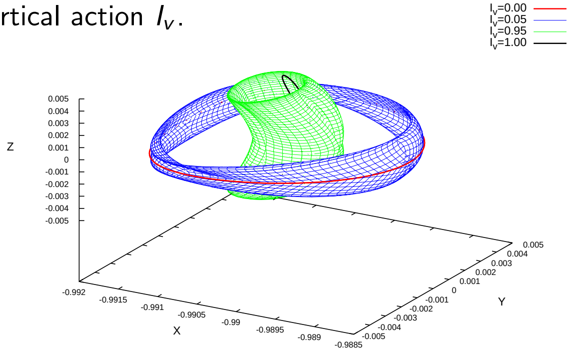


Figure: High energy level $C = 3.00083$

Transition Matrix

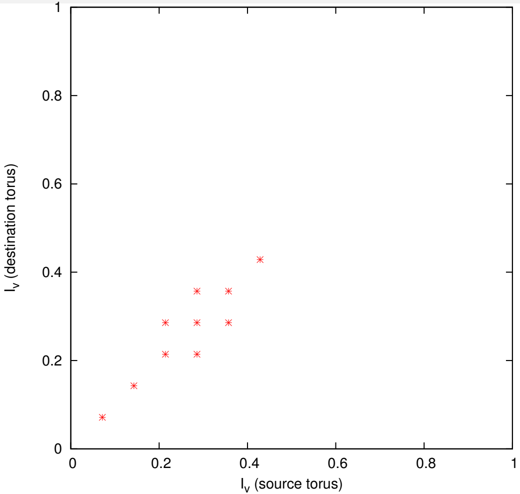


Figure: Low energy level $C = 3.00087$

Transition Matrix

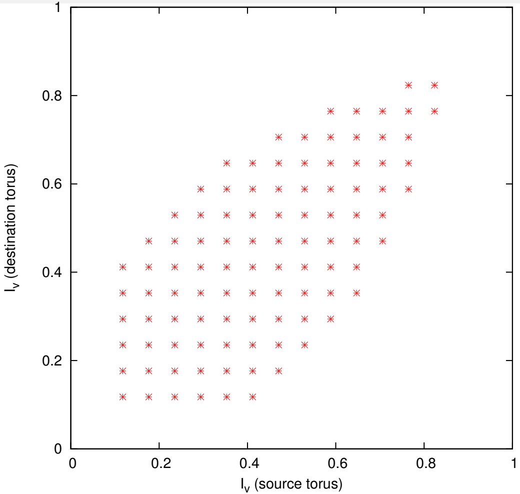


Figure: High energy level $C = 3.00083$

Main theoretical result (D-Gidea-Roldán 17)

Main Theorem. Given $\delta > 0$.
Assume $\exists \{\mathcal{L}^{\Sigma}_{I_j}\}_{j=0,N}$ level sets of I_v, with $0 < I_j < I_{max}$, and δ_j with $0 < \delta_j < \delta/2$, s.t., for each $j = 0, \ldots, N-1$:

(i) $\exists$ scattering map $\sigma^{\Sigma}_{i(j)}$ and pt. $(I_j, \phi_j) \in \mathcal{L}^{\Sigma}_{I_j}$ s.t. $B_{\delta_j}(I_j, \phi_j) \subset \mathrm{dom}\sigma^{\Sigma}_{i(j)}$,

(ii) $\exists k_j > 0$ s.t. $\mathrm{int}[F^{k_j} \circ \sigma^{\Sigma}_{i(j)}(B_{\delta_j}(I_j, \phi_j))] \supseteq B_{\delta_{j+1}}(I_{j+1}, \phi_{j+1})$

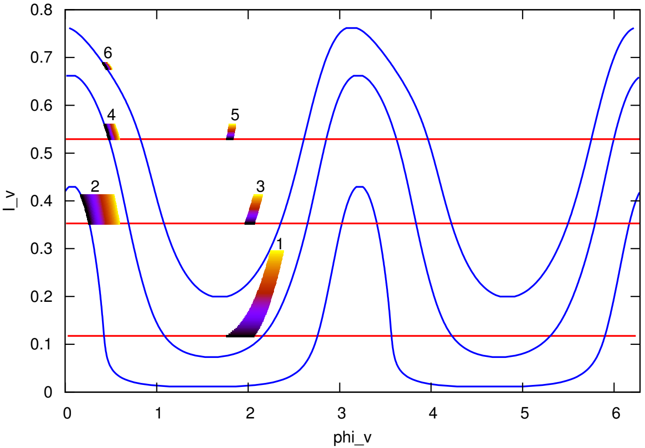
Then $\exists$ an orbit z_j of F in Σ, $j = 0, \ldots, N$, and a sequence of positive integers $n_j > 0$, $j = 0, \ldots, N-1$, such that $z_{j+1} = F^{n_j}(z_j)$ and

$$d(z_j, \mathcal{L}^{\Sigma}_{I_j}) < \delta/2, \text{ for all } j = 0, \ldots, N. \tag{19}$$

Consequently, there exist a trajectory $\Phi^t(z)$ of the Hamiltonian flow, and a finite sequence of times $0 = t_0 < t_1 < t_2 < \ldots < t_N$, such that

$$d(\Phi^{t_j}(z), \mathcal{L}_{I_j}) < \delta. \tag{20}$$

Main theoretical result



- Try to find drift orbits by constructing pseudo-orbits consisting of successive applications of several scattering maps
- Obtain theoretical results, using Hill locally and Kepler globally
- Add time dependent perturbation—elliptic orbit of Jupiter—and derive the existence of drift orbits