Aspectes d'integrabilitat per a algunes famílies paramètriques.

Chara Pantazi

Barcelona, 11 d'octubre de 2023

- Liouville (1835), A. Ostrowski(1946), Ritt(1948)
- R. H. Risch, The problem of integration in finite terms, *Trans. Amer. Math. Soc.*, 139, 167—189, 1969,
- M. Rosenlicht, Integration in finite terms, *Am. Math. Mon.*, 79, 963—972, 1972,
- M. Singer, Liouvillian first integrals of differential equations, *Trans. Amer. Math. Soc.*, 333(2), 673–688, 1992.
- Żołądek(1998), Olagnier(1996+), Casale(Malgrange) (2011), Zhang (2016).....

Two dimensions

Consider $X = (P, Q)$ with P,Q polynomials.

$$\dot{x} = P, \;\; \dot{y} = Q, \qquad\qquad y' = \frac{dy}{dx} = \frac{Q}{P}$$

- **Singer (1992)**
  X has a Liouvillian first integral if and only if has an integrating factor given as a product of polynomials and exponentials (Darboux functions).
  $$e^{g/f} \prod f_i^{\alpha_i}$$
- **Preller and Singer (1983)**. If X has an elementary first integral then the integrating factor is the N-th root of a rational function.
- X has a Darboux first integral if and only if has a rational integrating factor. (GGGL2003, CLPW2019)

Two dimensions

Go to Exemple

- A. Ferragut and H. Giacomini (2010) Algorithm for rational first integrals
- G. Chèze (2011) Algorithm for finding Darboux polynomials of given degree using the extactic curve.
- A. Ferragut and A. Gasull (2015), improved a method for calculate Darboux polynomials
- A. Bostan, G. Chèze, T. Cluzeau and J.A. Weil (2016) Algorithm for rational first integrals of given degree
- A. Ferragut, C. Galindo and F. Monserrat(2019) method for calculate special Darboux first integrals
- G. Chèze and T. Combot (2019), a better algorithm for calculation of Liouvilian first integrals using a more general concept of extactic curves.
- L.G.S.Duarte, L.A.C.P.daMota(2021), J. Avellar, M.S. Cardoso, L.G.S. Duarte, L.A.C.P. da Mota(2019),Combot(2019)....

Liouvillian Extension, Singer 1992

Definition-1 An extension $L \supset K$ of differential fields is a *Liouvillian extension of K* if $C_K = C_L$ and if there exists a tower of fields of the form

$$K = K_0 \subset K_1 \subset \ldots \subset K_i \subset K_{i+1} \ldots \subset K_m = L, \tag{1}$$

such that for each $i \in \{0, \ldots, m-1\}$ we have one of the following:

(i) $K_{i+1} = K_i(t_i)$, where $t_i \neq 0$ and $\underline{\partial t_i / t_i \in K_i}$ for all $\partial \in \Delta$;

(ii) $K_{i+1} = K_i(t_i)$, where $\underline{\partial t_i \in K_i}$ for all $\partial \in \Delta$;

(iii) $K_{i+1} = K_i(t_i)$, where t_i is finite algebraic over K_i.

If L is a differential extension of $K = \mathbb{C}(x_1, \ldots, x_n)$
L' the space of differential 1-forms with coefficients in L.

$$\alpha \in L' : \alpha = \sum a_i \, dx_i, \qquad a_i \in L$$

Liouvillian Extension, Singer 1992

Definition-1 An extension $L \supset K$ of differential fields is a *Liouvillian extension of K* if $C_K = C_L$ and if there exists a tower of fields of the form

$$K = K_0 \subset K_1 \subset \ldots \subset K_i \subset K_{i+1} \ldots \subset K_m = L, \tag{1}$$

such that for each $i \in \{0, \ldots, m-1\}$ we have one of the following:

(i) $K_{i+1} = K_i(t_i)$, where $t_i \neq 0$ and $\underline{\partial t_i / t_i \in K_i}$ for all $\partial \in \Delta$;

(ii) $K_{i+1} = K_i(t_i)$, where $\underline{\partial t_i \in K_i}$ for all $\partial \in \Delta$;

(iii) $K_{i+1} = K_i(t_i)$, where t_i is finite algebraic over K_i.

If L is a differential extension of $K = \mathbb{C}(x_1, \ldots, x_n)$
L' the space of differential 1-forms with coefficients in L.

$$\alpha \in L' : \alpha = \sum a_i \, dx_i, \qquad a_i \in L$$

Using 1-forms

Remark-1 L is a differential extension of $K = \mathbb{C}(x_1, \ldots, x_n)$,
we can restate conditions (i)–(iii) in Definition 1:

(i) $K_{i+1} = K_i(t_i)$, where $t_i \neq 0$ and $\underline{dt_i = \delta_i t_i}$ with some $\delta_i \in K_i'$ (necessarily $d\delta_i = 0$).

(ii) $K_{i+1} = K_i(t_i)$, where $\underline{dt_i = \delta_i}$ with $\delta_i \in K_i'$ (necessarily $d\delta_i = 0$).

(iii) K_{i+1} is a finite algebraic extension of K_i.

Darboux functions. Why we use them?

Darboux functions are of the form

$$\phi = \exp(g/f) \prod f_i^{a_i}, \tag{2}$$

f_i and g and f are elements of $\mathbf{C}[x_1, \ldots, x_n]$
a_i are complex numbers.
Given Darboux function $\phi \Rightarrow d\phi/\phi$, is clearly a closed rational 1-form.
Conversely
Every closed rational 1-form must be the logarithmic differential of some Darboux function.

Theorem

Consider a 1–form $\alpha \in \mathbb{C}(x_1, \ldots, x_n)'$. If α is closed, then there exist elements $g, f, f_i \in \mathbb{C}[x_1, \ldots, x_n]$ and constants $a_i \in \mathbf{C}$ such that

$$\alpha = d\left(\frac{g}{f}\right) + \sum a_i \frac{df_i}{f_i}.$$

Go to Exemple 1

Darboux functions. Why we use them?

Darboux functions are of the form

$$\phi = \exp(g/f)\prod f_i^{a_i}, \tag{2}$$

f_i and g and f are elements of $\mathbf{C}[x_1, \dots, x_n]$
a_i are complex numbers.
Given Darboux function $\phi \Rightarrow d\phi/\phi$, is clearly a closed rational 1-form.
Conversely
Every closed rational 1-form must be the logarithmic differential of some Darboux function.

Theorem

Consider a 1–form $\alpha \in \mathbb{C}(x_1, \dots, x_n)'$. If α is closed, then there exist elements $g, f, f_i \in \mathbb{C}[x_1, \dots, x_n]$ and constants $a_i \in \mathbf{C}$ such that

$$\alpha = d\left(\frac{g}{f}\right) + \sum a_i \frac{df_i}{f_i}.$$

Go to Exemple 1

Darboux functions. Why we use them?

Darboux functions are of the form

$$\phi = \exp(g/f) \prod f_i^{a_i}, \tag{2}$$

f_i and g and f are elements of $\mathbf{C}[x_1, \ldots, x_n]$
a_i are complex numbers.
Given Darboux function $\phi \Rightarrow d\phi/\phi$, is clearly a closed rational 1-form.
Conversely
Every closed rational 1-form must be the logarithmic differential of some Darboux function.

Theorem

Consider a 1–form $\alpha \in \mathbb{C}(x_1, \ldots, x_n)'$. If α is closed, then there exist elements $g, f, f_i \in \mathbb{C}[x_1, \ldots, x_n]$ and constants $a_i \in \mathbf{C}$ such that

$$\alpha = d\left(\frac{g}{f}\right) + \sum a_i \frac{df_i}{f_i}.$$

Go to Exemple 1

Darboux functions. Why we use them?

Darboux functions are of the form

$$\phi = \exp(g/f) \prod f_i^{a_i}, \tag{2}$$

f_i and g and f are elements of $\mathbf{C}[x_1, \ldots, x_n]$
a_i are complex numbers.
Given Darboux function $\phi \Rightarrow d\phi/\phi$, is clearly a closed rational 1-form.
Conversely
Every closed rational 1-form must be the logarithmic differential of some Darboux function.

Theorem

Consider a 1–form $\alpha \in \mathbb{C}(x_1, \ldots, x_n)'$. If α is closed, then there exist elements $g, f, f_i \in \mathbb{C}[x_1, \ldots, x_n]$ and constants $a_i \in \mathbf{C}$ such that

$$\alpha = d\left(\frac{g}{f}\right) + \sum a_i \frac{df_i}{f_i}.$$

Go to Exemple 1

Liouvillian integrable one form

Definition 2 Given a 1-form $\omega \in \mathbb{C}(x_1, \ldots, x_n)'$, we say that ω is *Liouvillian integrable* if there exists a 1-form $\alpha \in L'$ for some Liouvillian extension L of $\mathbb{C}(x_1, \ldots, x_n)$ such that

$$
\begin{aligned}
d\omega = \quad & \alpha \wedge \omega \\
& d\alpha = 0
\end{aligned}
$$

.

Remark

(a) We note:
- If $\exists\phi$ in some L of $\mathbb{C}(x_1, \ldots, x_n)$ s.t.

$$d\phi \wedge \omega = 0$$

then

$$\omega = m\, d\phi \qquad m \in L \Rightarrow d\omega = \frac{1}{m} dm \wedge \omega = \alpha \wedge \omega,$$

- Conversely, if

$$d\omega = \alpha \wedge \omega \quad d\alpha = 0$$

by Remark 1(i), there exists m in a Liouvillian extension L_1 of L such that $\alpha = -dm/m$, whence

$$d(m\omega) = dm \wedge \omega + md\omega = m(-\alpha \wedge \omega + d\omega) = 0,$$

which implies $m\omega = d\phi$ for some ϕ in a Liouvillian extension $L_2 \supset L_1$, hence of $\mathbb{C}(x_1, \ldots, x_n)$.

We call m an **inverse integrating factor** for ω.

(b) In particular, Definition 2 implies that $d\omega \wedge \omega = 0$, so that ω is completely integrable in the usual sense (cf. Camacho and Lins Neto).

Theorem (Singer's Theorem for 1-forms)

Let ω be a rational 1–form over $\mathbb{C}(x_1, \dots, x_n)$. Then ω is Liouvillian integrable if and only if there exists a closed 1–form $\underline{\alpha \in \mathbb{C}(x_1, \dots, x_n)'}$ *s.t.*

$$d\omega = \alpha \wedge \omega, \qquad d\alpha = 0.$$

M. Singer, *Liouvillian first integrals of differential equations*, *Trans. Amer. Math. Soc.*, 333(2), 673–688, 1992. ▸ Go to Exemple 2 ▸ Go to Exemple 3

Combining Theorem 1 and Theorem 2, we see that a 1-form ω is Liouvillian integrable if and only if it admits a Darboux integrating factor.

We will consider three-dimensional rational vector fields

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z} \quad (3)$$

in $\mathbb{C}^3$; equivalently we will look at the corresponding 2-forms

$$\Omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy. \quad (4)$$

Definition

A non-constant element, ϕ, of a Liouvillian extension of $\mathbb{C}(x, y, z)$ is called a Liouvillian first integral of the vector field $\mathcal{X}$ if it satisfies $\mathcal{X}\phi = 0$ or, equivalently, $d\phi \wedge \Omega = 0$.

Theorem (First extension of Singer's theorem for 2-forms in three dimensions)

Let $K = \mathbf{C}(x, y, z)$, and let Ω be our 2–form. If there exists a Liouvillian first integral of Ω,
then one of the following holds:

(I) *There exist 1–forms $\underline{\omega, \alpha \in K'}$ such that*

$$\omega \neq 0,\ \omega \wedge \Omega = 0,\ \alpha \wedge \omega = d\omega,\ d\alpha = 0.$$

So, Ω is Liouvillian integrable over K.

(II) *There exists a $\underline{\text{1–form } \beta \in K'}$ such that $\beta \wedge \Omega = d\Omega$ with $d\beta = 0$. So, Ω admits an inverse Jacobi multiplier of Darboux type over $K = \mathbf{C}(x, y, z)$.*

▸ Go to Exemple 4 ▸ Go to Exemple 5

Remark

(a) Roughly speaking, condition I means there is a first integral of the form $\phi = \int \frac{\omega}{e^{\int \alpha}}$. Note that $e^{\int \alpha}$ is of Darboux type. In the special case when $\alpha = 0$, there is a first integral of the form $\int \omega$.

(b) In the same way, condition II means that Ω admits an inverse Jacobi multiplier of the form $e^{\int \beta}$, with $\beta \in K'$, i.e. of Darboux type.

Remark So there is some Liouvillian extension L of K and one-forms $\omega \neq 0$, α in L' such that

$$\omega \wedge \Omega = 0, \quad d\omega = \alpha \wedge \omega, \quad d\alpha = 0. \tag{5}$$

We will briefly say that Ω is Liouvillian integrable.

About the proof

Induction on the tower of fields

$$K = K_0 \subset K_1 \subset \ldots \subset K_i \subset K_{i+1} \ldots \subset K_m = L,$$

Notation:I_i, II_i on K_i

Claim-1: If condition II_{i+1} holds then condition II_i or condition I_i holds.

Claim-2: If condition I_{i+1} then one of conditions II_i, I_i must hold.

I_{i+1} always imply I_i, unless Ω admits two independent Liouvillian first integrals.

About the proof

Induction on the tower of fields

$$K = K_0 \subset K_1 \subset \ldots \subset K_i \subset K_{i+1} \ldots \subset K_m = L,$$

Notation: I_i, II_i on K_i

Claim-1: If condition II_{i+1} holds then condition II_i or condition I_i holds.

Claim-2: If condition I_{i+1} then one of conditions II_i, I_i must hold.

I_{i+1} always imply I_i, unless Ω admits two independent Liouvillian first integrals.

Corollary

Let $K = \mathbf{C}(x,y,z)$, and let Ω be the 2–form (4) over K. If Ω admits a Liouvillian first integral, but not two independent Liouvillian first integrals, then Ω is Liouvillian integrable over K.

Corollary

Let $K = \mathbf{C}(x,y,z)$ and let Ω be the 2–form (4) over K. Assume that Ω does not admit a Liouvillian first integral, but there exists a Liouvillian extension L of K and a 1–form $\beta \in L'$ such that $\beta \wedge \Omega = d\Omega$ with $d\beta = 0$. Then there exists a 1–form $\overline{\beta} \in K'$ such that $\overline{\beta} \wedge \Omega = d\Omega$ with $d\overline{\beta} = 0$.

Theorem (Second extension of Singer's theorem for 2-forms in three dimensions)

Let $K = \mathbf{C}(x, y, z)$, and let Ω be the 2–form (4) over K. If there exists a Liouvillian first integral of Ω, then there exists a finite algebraic extension $\widetilde{K}$ of K such that Ω is Liouvillian integrable over $\widetilde{K}$.

If Ω has a Liouvillian first integral then it will admit one defined in some algebraic extension of the rational functions.

Theorem (Second extension of Singer's theorem for 2-forms in three dimensions)

Let $K = \mathbf{C}(x, y, z)$, and let Ω be the 2–form (4) over K. If there exists a Liouvillian first integral of Ω, then there exists a finite algebraic extension $\widetilde{K}$ of K such that Ω is Liouvillian integrable over $\widetilde{K}$.

If Ω has a Liouvillian first integral then it will admit one defined in some algebraic extension of the rational functions.

$K \subset L_0 \subset L_0(t) \subset L_0(t,q) = L$

Lemma

Let L_0 *be a Liouvillian extension of* $K = \mathbb{C}(x,y,z)$,
t *transcendental over* L_0,
and q *algebraic over* $L_0(t)$,
thus $L := L_0(t,q)$ *Liouvillian over* L_0.
Moreover let $\omega, \alpha \in L'$ *such that* $\omega \neq 0$, $\omega \wedge \Omega = 0$, $d\omega = \alpha \wedge \omega$, $d\alpha = 0$.

Then there exists a finite algebraic extension $\widetilde{L}_0$ *of* L_0, *and* $\widetilde{\omega}, \widetilde{\alpha} \in \widetilde{L}'_0$ *such that* $\widetilde{\omega} \neq 0$, $\widetilde{\omega} \wedge \Omega = 0$, $d\widetilde{\omega} = \widetilde{\alpha} \wedge \widetilde{\omega}$, $d\widetilde{\alpha} = 0$.

Briefly, Ω *is Liouvillian integrable over* $\widetilde{L}_0$.

Proof of Theorem:

$$K = K_0 \subset K_1 \subset K_{i-1} \subset K_i \subset K_{i+1} \ldots \subset K_m = L,$$

and assume that for some $i > 1$ one has a finite algebraic extension $K_{i+1} \supset K_i$, and $\omega, \alpha \in K'_{i+1}$ and

$$\omega \neq 0, \, \omega \wedge \Omega = 0, \; \alpha \wedge \omega = d\omega, \; d\alpha = 0.$$

With no loss of generality, $K_i \supset K_{i-1}$ is then transcendental, and Lemma 7 shows that there exists a finite algebraic extension $\widetilde{K}_{i-1}$ of K_{i-1}, and $\widetilde{\omega}, \widetilde{\alpha} \in K'_{i-1}$ as required. Thus all transcendental extensions can be eliminated by descent.

Conclusions: Singer Theorem for 3 dimensions

- There is either
  - (I) A first integral whose differential is the product of a Darboux function with a rational 1-form, or
  - (II) There exists an inverse Jacobi multiplier of Darboux type.
- If there is only one Liouvillian first integral then the first alternative holds.
- In any case there exists a first integral that is defined over a finite algebraic extension L of the rational function field $K = C(x, y, z)$.

Future work

- whether there indeed exist cases with $L = K(x, y, z)$ and what such cases look like.
- Several criteria which guarantee that $L = K$ can be chosen.
- ....

N=3, Exemple: A simplified Laser system

$$\begin{aligned} \dot{x} &= xz + y + a, \\ \dot{y} &= yz - x, \\ \dot{z} &= b - x^2 - y^2, \end{aligned} \tag{6}$$

- A. Politi, G.L. Oppo & R. Badii, *Phys. Rev. A*, **33**, (1986) 4055.
- F.T. Arecchi, G.L. Lippi, G.P. Puccioni & J.R. Tredicce, *Opt. Commun.*, **51**, (1984) 308–314.
- Y. Li, M. Yuan & Z. Chen, *Chaos, Solit. & Fractals*, **159**, (2022) 112114.

Theorem

The simplified Laser system for $a = 0$, is completely integrable with the following two functionally independent first integrals

(a) *If $a = 0$ and $b \neq 0$*

$$H_1(x,y,z) = \frac{\left(x^2+y^2\right)^b}{e^{x^2+y^2+z^2}},$$

$$H_2(x,y,z) = 2\arctan\left(\frac{x}{y}\right) - \int_0^{x^2+y^2} \frac{1}{s\sqrt{b\ln(s) - \ln\left(e^{-x^2-y^2-z^2}\left(x^2+y^2\right)^b\right.}}$$

(b) *For $a = b = 0$*

$$H_1(x,y,z) = x^2+y^2+z^2,$$

$$H_2(x,y,z) = \frac{\exp\left(2\sqrt{x^2+y^2+z^2}\arctan\left(\frac{y}{x}\right)\right)\left(\sqrt{x^2+y^2+z^2}-z\right)^2}{x^2+y^2}.$$

Exemple: (2-dimensions) The Selkov system

$$\begin{aligned} \dot{x} &= 1 - xy^n = P, \\ \dot{y} &= ay(-1 + xy^{n-1}) = Q, \end{aligned}$$

where n is a positive integer and $a > 0$.

- E.E. Selkov, *Self-oscillations in glycolysis*, I. A simple kinetic model. Eur. J. Biochem. **4** (1968) 79–86.
- J. Higgins, A chemical mechanism for oscillation of glycolytic intermediates in yeast cells, Proc. Natl. Acad. Sci. (USA) **51** (1964) 989–994.

One way: Singer's Theorem

Theorem

Selkov's system for $a > 0$ is not Liouvillian integrable.

Lemma

The unique irreducible invariant algebraic curve of Selkov's system is $y = 0$.

Lemma The invariant line at infinity has multiplicity $n+1$. The only exponential factors of Selkov's system are $G_i = \exp\left((ax+y)^i\right)$ for $i = 1, \cdots, n$.

One way: Singer's Theorem

Theorem

Selkov's system for $a > 0$ is not Liouvillian integrable.

Lemma

The unique irreducible invariant algebraic curve of Selkov's system is $y = 0$.

Lemma The invariant line at infinity has multiplicity $n + 1$. The only exponential factors of Selkov's system are $G_i = \exp\left((ax + y)^i\right)$ for $i = 1, \cdots, n$.

One way: Singer's Theorem

Theorem

Selkov's system for $a > 0$ is not Liouvillian integrable.

Lemma

The unique irreducible invariant algebraic curve of Selkov's system is $y = 0$.

Lemma The invariant line at infinity has multiplicity $n + 1$. The only exponential factors of Selkov's system are $G_i = \exp\left((ax + y)^i\right)$ for $i = 1, \cdots, n$.

Another way: Morales-Ramis-Simó

- J.J. Morales-Ruiz, J.P. Ramis, C. Simó, *Integrability of Hamiltonian Systems and Differential Galois Groups of Higher Variational Equations*, Ann. Sc. École Norm. Sup. **40** (2007) 845–884.
- M. Ayoul, N.T. Zung, *Galoisian obstructions to non-Hamiltonian integrability*, C. R. Math. Acad. Sci. Paris, **348** (2010) 1323–1326.
- Acosta-Humánez, P.B and Lázaro, J.T and Morales-Ruiz, J.J. and Pantazi, C. *Differential Galois theory and non-integrability of planar polynomial vector fields*, J.D.E. **264** (2018), 7183–7212

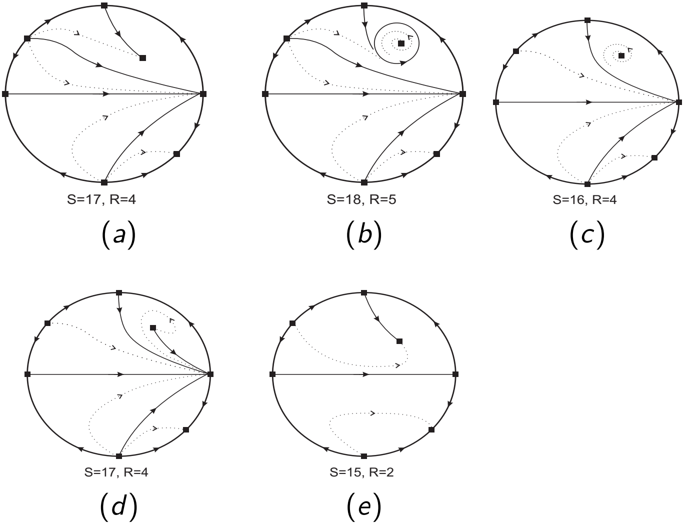

Figure: 1. The phase portraits of Selkov systems for n odd, $n \geq 1$.

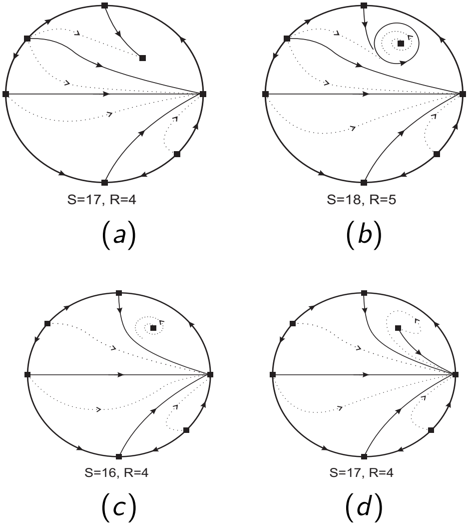

Figure: 2. The phase portraits of Selkov's systems for n even, $n \geq 2$.

Theorem

For the Selkov's system with $a > 0$ and $n \geq 1$, there are exactly nine non–topological equivalent phase portraits in the Poincaré disc:

(a) *For $n = 1$ see Figure 1(e) with $S = 15$ and $R = 2$.*

(b) *For $n \geq 2$ and n odd (resp. even)*

   (i) *Figure 1(a) (resp. 2(a)) with $a \in (0,\ 1/(n-1))$ and $S = 17$ and $R = 4$.*

   (ii) *Figure 1(b)(resp. 2(b)) with $a \in (1/(n-1),\ a^*)$ and a^* is a unique constant in the interval $\left(\frac{1}{n-1},\ \frac{2^n-1}{2^n-2}\right)$. In this case, $S = 18$ and $R = 5$.*

   (iii) *Figure 1(c) (resp. 2(c)) with $a = a^*$ and $S = 16$ and $R = 4$.*

   (iv) *Figure 1(d) (resp. 2(d)) with $a > a^*$ and $S = 17$ and $R = 4$.*

Theorem

Go Back *For every positive integer* $n \geq 2$*, there exists a unique constant* $a^* \in \left(\frac{1}{n-1}, \frac{2^n-1}{2^n-2}\right)$ *such that Selkov's system has no periodic orbits when* $a \in (-\infty, 1/(n-1)] \bigcup [a^*, +\infty)$ *and has a unique limit cycle when* $a \in (1/(n-1), a^*)$*, which is stable and hyperbolic. Moreover, when the limit cycle exists, its amplitude increases with* a*.*

- H. Chen, J. Llibre and Y. Tang, J. Nonlinear Sci. **31** 85 (2021) 25 pages.
- H. Chen and Y. Tang, J. Differential Equations **266** (2019) 7638–7657.
- J.C. Artés, J. Llibre and C. Valls, Chaos.Sol.Frac. **114** (2018) 145–150.
- J. Llibre and M. Mousavi, Discrete Contin. Dyn. Syst. Ser. B, **27** (2021) 245–256.

Exemple 1

$n = 4$. Consider the closed one form

$$\alpha = \left(-\frac{2w}{x^3} + z + \frac{y^2}{2} + \frac{1}{x}\right) dx + (yx + w + z)dy + \left(-\frac{w}{z^2} + x + y + z\right) dz + \left(\frac{1}{x^2} + y + \frac{1}{z} + \frac{1}{w}\right) dw$$

Then, according to Theorem 1 there exists $f_1, f_2, f, g \in \mathbf{C}[x, y, z, w]$ and $a_1, a_2 \in \mathbf{C}$ such that

$$\alpha = a_1 \frac{df_1}{f_1} + a_2 \frac{df_2}{f_2} + d\left(\frac{g}{f}\right).$$

Take $a_1 = a_2 = 1$,
$f_1 = x,\ f_2 = w, f = 2zx^2$ and
$g = x^3 z\, y^2 + 2y\, x^2 zw + 2x^3 z^2 + 2x^2 z^2 y + x^2 z^3 + 2x^2 w + 2zw.$ Go Back

Exemple 1

$n = 4$. Consider the closed one form

$$\alpha = \left(-\frac{2w}{x^3} + z + \frac{y^2}{2} + \frac{1}{x}\right) dx + (yx + w + z) dy + \left(-\frac{w}{z^2} + x + y + z\right) dz + \left(\frac{1}{x^2} + y + \frac{1}{z} + \frac{1}{w}\right) dw$$

Then, according to Theorem 1 there exists $f_1, f_2, f, g \in \mathbf{C}[x, y, z, w]$ and $a_1, a_2 \in \mathbf{C}$ such that

$$\alpha = a_1 \frac{df_1}{f_1} + a_2 \frac{df_2}{f_2} + d\left(\frac{g}{f}\right).$$

Take $a_1 = a_2 = 1$,
$f_1 = x,\ f_2 = w, f = 2zx^2$ and
$g = x^3 z\, y^2 + 2y\, x^2 zw + 2x^3 z^2 + 2x^2 z^2 y + x^2 z^3 + 2x^2 w + 2zw.$ Go Back

Exemple 2

$n = 3$. Consider the one form

$$\omega = (x^3 + xy - 1)dx + (-xy^2 + x^2 + xz)dy + xydz.$$

Note that

$$d\omega = (-y^2 + x + z)dx \wedge dy + ydx \wedge dz = \alpha \wedge \omega, \qquad d\alpha = 0$$

$$\alpha = (x^2 + y)dx + (-y^2 + x + z)dy + ydz.$$

From Theorem 2 we have that ω is Liouvillian integrable. There is $m \in L_1$, s.t. $\alpha = -dm/m$.

$$m = \mathrm{e}^{\frac{y^3 - 3xy - 3zy - x^3}{3}}$$

and satisfy $d(m\omega) = 0$. Then, there is Φ in some $L_2 \supseteq L_1$ such that $m\omega = d\Phi$.

$$\Phi = -x\mathrm{e}^{\frac{y^3 - 3xy - 3zy - x^3}{3}}.$$

Here $L_2 = L_1$.

Go Back

Exemple 3

Consider

$$\omega = \frac{-x^6yz + yzx^4 + x^5 + x^2yz - 2\,x^3 - 3\,yz + x}{(x^2-1)^2\,zx}dx + \frac{1}{x^2-1}dy + \frac{1}{x^3z^2}dz.$$

Note that

$$d\omega = \frac{x^4-3}{x\,(x^2-1)}dx\wedge dy + \frac{x^4-3}{x^4z^2}dx\wedge dz = \alpha\wedge\omega$$

with $\alpha = (x^4-3)/x\,dx$ and $d\alpha = 0$ so ω L. I. There is $m \in L_1$, such that $\alpha = -dm/m$.

$$m = x^3\mathrm{e}^{-1/4\,x^4}$$

and satisfy $d(m\omega) = 0$. There is Φ in $L_2 \subseteq L_1$ s.t. $m\omega = d\Phi$.

$$\Phi = \frac{\mathrm{e}^{-1/4\,x^4}\left(x^3yz - x^2 + 1\right)}{z\,(x^2-1)}.$$

Here $L_2 = L_1$. Go Back

Exemple 4

Consider the vector field

$$\begin{aligned} \dot{x} &= -144x\,(x-1)\,, \\ \dot{y} &= 144x^2y^2 - 144x\,y^2 + 168yx - 96y - 3, \\ \dot{z} &= z, \end{aligned} \tag{7}$$

and we obtain the Riccati equation

$$\frac{dy}{dx} = -y^2 - \frac{(7x-4)}{6x\,(x-1)}y + \frac{1}{48x\,(x-1)}.$$

Under the change of variable $y(x) = \xi'(x)/\xi(x)$ we obtain the second order differential equation (9) which has two algebraic solutions
System (7) admits the invariant polynomials

$$f_1 = 6912x^4y^4 - 13824x^3y^4 + 6912x^2y^4 - 288x^2y^2 + 288x\,y^2 + 32xy - 32y - 1$$

$$\begin{aligned} f_2 &= 20736x^5y^4 - 41472x^4y^4 - 6912x^4y^3 + 20736x^3y^4 + 13824x^3y^3 + 864x^3y^2 \\ &\quad -6912x^2y^3 - 864x^2y^2 - 48x^2y + 48xy + x - 4 \end{aligned}$$

with cofactors $576x^2y - 576xy + 96x - 96$ and $576x(x-1)y$ respectively. It turns out that system (7) has the two rational first integrals,

$$H_1 = \frac{x^2 f_1^3}{f_2^3}, \qquad H_2 = \frac{z^{144}(x-1)}{x}.$$

The corresponding two form of system (7) is

$$\Omega = -144x(x-1)\,dy \wedge dz + (144x^2y^2 - 144x\,y^2 + 168yx - 96y - 3)dz \wedge dx + zdx$$

Consider

$$\omega = \left(48x^2y^2 - 48x\,y^2 + 56yx - 32y - 1\right)dx + \left(48x^2 - 48x\right)dy.$$

Then $\omega \wedge \Omega = 0$. Now consider the closed one form α (is rational) (the expression is very large).

$$\omega \wedge \Omega = 0, \alpha \wedge \omega = d\omega, d\alpha = 0.$$

Go Back

Exemple 5

$$
\begin{aligned}
\dot{x} &= 1\\
\dot{y} &= z\\
\dot{z} &= -\frac{(7x-4)}{6x\,(x-1)}z + \frac{1}{48x\,(x-1)}y
\end{aligned}
\tag{8}
$$

Consider $\xi = y$. Then from system (8) since $dt = dx$ we have $\dot{\xi} = z$ and so arises s.ode

$$
\frac{d^2}{dx^2}\xi(x) + \frac{(7x-4)}{6x\,(x-1)}\frac{d}{dx}\xi(x) - \frac{\xi(x)}{48x\,(x-1)} = 0 \tag{9}
$$

and has the two algebraic solutions

$$
\xi_1(x) = \left((-x)^{\frac{1}{3}}+1\right)^{\frac{1}{4}}\left(\frac{\sqrt{3}\,(-x)^{\frac{1}{3}}-\sqrt{3}+2\sqrt{(-x)^{\frac{2}{3}}-(-x)^{\frac{1}{3}}+1}}{\sqrt{3}\,(-x)^{\frac{1}{3}}-\sqrt{3}-2\sqrt{(-x)^{\frac{2}{3}}-(-x)^{\frac{1}{3}}+1}}\right)^{\frac{1}{8}}
$$

$$
\xi_2(x) = \frac{\left((-x)^{\frac{1}{3}}+1\right)^{\frac{1}{4}}\left(-\sqrt{3}\,(-x)^{\frac{1}{3}}+\sqrt{3}+2\sqrt{(-x)^{\frac{2}{3}}-(-x)^{\frac{1}{3}}+1}\right)^{\frac{1}{8}}}{\left(\sqrt{3}\,(-x)^{\frac{1}{3}}-\sqrt{3}-2\sqrt{(-x)^{\frac{2}{3}}-(-x)^{\frac{1}{3}}+1}\right)^{\frac{1}{8}}}.
$$

$$f_1(x,y,z) = y\frac{d\xi_1(x)}{dx} - z\xi_1(x) = y\xi_1'(x) - z\xi_1(x) = 0$$
$$f_2(x,y,z) = y\frac{d\xi_2(x)}{dx} - z\xi_2(x) = y\xi_2'(x) - z\xi_2(x) = 0$$

are invariant for system (8) with the same rational cofactor

$$K_1(x) = K_2(x) = -\frac{7x-4}{6x\,(x-1)}$$

and so we obtain the first integral (non constant)

$$\Phi(x,y,z) = \frac{f_1(x,y,z)}{f_2(x,y,z)} = \frac{y\xi_1'(x) - z\xi_1(x)}{y\xi_2'(x) - z\xi_2(x)}.$$

$$\omega = \left(-\frac{1}{48x\,(x-1)}\xi_1 y + z\xi_1' + \frac{7x-4}{6x\,(x-1)}\xi_1 z\right)dx + \xi_1' dy - \xi_1 dz,$$

$$\alpha = -\frac{(7x-4)}{6x\,(x-1)}dx$$

$\omega \wedge \Omega = 0$, $d\omega = \alpha \wedge \omega$ and $d\alpha = 0$. Go Back