On the escaping set of transcendental maps

(From the exponential family to Eremenko's conjecture)

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The escaping set

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire (polynomial or transcendental) map of degree $d \geq 2$.

Definition: The escaping set, $\mathcal{I}(f)$, is defined as the set of points whose orbit escape to infinity under iteration. That is,

$$\mathcal{I}(f) = \{z \in \mathbb{C} \mid f^n(z) \to \infty\}.$$

Examples.

- If $f(z) = z^2$ then $0 \notin \mathcal{I}(f)$ while $2 \in \mathcal{I}(f)$.
- If $f(z) = \exp(z)$ then $0 \in \mathcal{I}(f)$.
- If $f(z) = 2\pi i \exp(z)$ then $0 \notin \mathcal{I}(f)$.

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Questions.

- Is $\mathcal{I}(f)$ always non-empty?
- Why $\mathcal{I}(f)$ is relevant in complex dynamics?
- What kind of topological/geometrical/analytic properties it might have?
- What about $HD(\mathcal{I}(f))$ (measure theoretical approach)?

The escaping set in transcendental dynamics

Walter Bergweiler and Lasse Rempe

June 23, 2025

Abstract

The escaping set of an entire function consists of the points in the complex plane which tend to infinity under iteration. This set plays a central role in the dynamics of transcendental entire functions. The goal of this survey is to explain this role, to summarise some of the main results in the area, and to identify a number of open questions.

(130 pages)

The Julia and Fatou sets

Definition. Let $\mathcal{F} := \{f_\lambda : U \to \mathbb{C}\}$ be family of holomorphic maps. Then $\mathcal{F}$ is normal if (all) sequences have partial convergence subsequences (equi-continuity).

Theorem (P. Montel, 1876-1975). If a family $\mathcal{F}$ omits two values $z_1 \neq z_2 \in \mathbb{C}$, then $\mathcal{F}$ is a normal family in U.

Examples. Let $f(z) = z^2$.

- $\mathcal{F}_1 := \{f^n : \mathbb{D} \to \mathbb{C}\}_{n\geq 1}$ is normal.
- $\mathcal{F}_1 := \{f^n : D_\epsilon(1) \to \mathbb{C}\}_{n\geq 1}$ is not normal.

The Julia and Fatou sets

Definition. Let $\mathcal{F} := \{f_\lambda : U \to \mathbb{C}\}$ be family of holomorphic maps. Then $\mathcal{F}$ is normal if (all) sequences have partial convergence subsequences (equi-continuity).

Definition. The Julia set of f, $\mathcal{J}(f)$, is the set of points $z \in \mathbb{C}$ for which the family of iterates $\{f^n(z)\}_{n\in\mathbb{N}}$ fails to be a normal family in any neighborhood U of z. CLOSED SET

Definition. The Fatou set, $\mathcal{F}(f)$, is defined as $\mathbb{C} \setminus \mathcal{J}(f)$. OPEN SET (connected components)

Remark. Complex dynamics is about the understanding of these two totally invariant sets (dynamical plane) and how these sets vary when we move the parameters defining the family under study (parameter plane/space).

The polynomial case: The basin of attraction of infinity

Let $P(z) = z^d + a_{d-1}z^{d-1} + \ldots + a_1 z + a_0,\ a_j \in \mathbb{C},\ d \geq 2.$

There exists $R > 0$ (big enough) such that $D(0, R) \subset \mathcal{I}(P)$. In fact, $z = \infty$ is a (super)attracting fixed point and $\mathcal{I}(P)$ coincides with the immediate basin of attraction of $z = \infty$

$$A^\star(\infty) = \{z \in \hat{\mathbb{C}} \mid P^n(z) \to \infty\} \quad (= A(\infty))$$

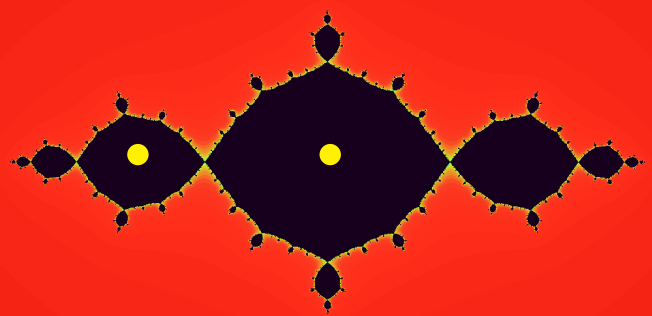
So, $\mathcal{I}(P) = A^\star(\infty) \subset \mathcal{F}(P)$ is the (open) Fatou domain associated to the attracting fixed point at infinity.

Remak. The topological properties (connectivity) of $A^\star(\infty)$ depends on the dynamical behaviour of the critical points (zeros of P').

The quadratic family: $Q_c(z) = z^2 + c,\ c \in \mathbb{C}$

$\mathcal{I}(P) = A^\star(\infty) \neq \emptyset \subset \mathcal{F}(P)$

$\mathcal{J}(P) = \partial\mathcal{I}(P)$

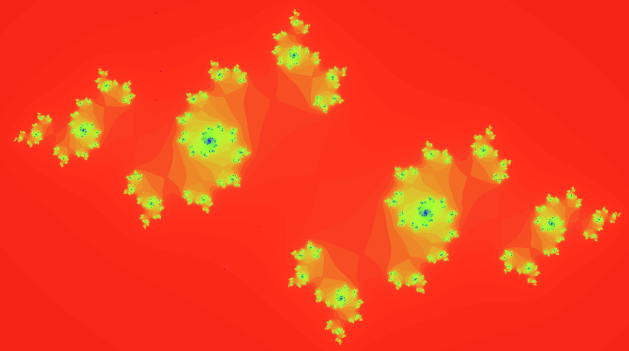


Basilica (connected)

$c = -1$

$\mathcal{I}(P) = A^\star(\infty) \neq \emptyset \subset \mathcal{F}(P)$

$\mathcal{J}(P) = \partial\mathcal{I}(P)$



Cantor (totally disconneted)

$|c| > 2$

No more discussions for $f = P$ polynomial.

The set $\mathcal{I}(f)$ in the entire transcendental case

Theorem (Charles E. Picard, 1856-1941). Let f be an entire transcendental map. Then any neighbourhood of $z = \infty$ (essential singularity) maps to $\mathbb{C}$, with the exception of at most one point (omitted value).

Theorem (A. Eremenko and P. Dominguez) Let f be an entire transcendental map. Then

(a) $\mathcal{I}(f) \neq \emptyset$ (and so, $\#\,\mathcal{I}(f) = \infty$).

(b) $\mathcal{J}(f) = \partial\mathcal{I}(f)$ (and so, $\mathcal{J}(f) \neq \emptyset$). [$\mathcal{I}(f)$ is relevant in complex dynamics]

(c) $\mathcal{I}(f) \cap \mathcal{J}(f) \neq \emptyset$. [$\mathcal{I}(f) \subset \mathcal{J}(f)$ in some cases]

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Proof of (b) assuming (a).

$\mathcal{J}(f) \subset \partial\mathcal{I}(f)$. Let $z \in \mathcal{J}(f)$ and consider V a neighborhood of z. The family $\{f^n|_V\}_{n\geq 0}$ is not normal.
Let $z_1 \in \mathcal{I}(f)$ and $z_2 = f(z_1) \neq z_1$. Hence, by Montel's Theorem, there exists $w \in V$ such that $(f^m(w) - z_1)(f^m(w) - z_2) = 0$, for some $m \geq 1$. We conclude $w \in (\mathcal{I}(f) \cap V)$. Clearly $\mathrm{Int}(\mathcal{I}(f)) \subset \mathcal{F}(f)$, so $z \in \partial\mathcal{I}(f)$.

$\partial\mathcal{I}(f) \subset \mathcal{J}(f)$. Similar (exercise).

The transcendental entire case

Theorem. Let f be an entire transcendental map. Then

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(b) $\mathcal{J}(f) = \partial\mathcal{I}(f)$ (and so, $\mathcal{J}(f) \neq \emptyset$).

(c) $\mathcal{I}(f) \cap \mathcal{J}(f) \neq \emptyset$.

Proof of (a) for f transcendental. Much more elaborate than for polynomials (or rationals).

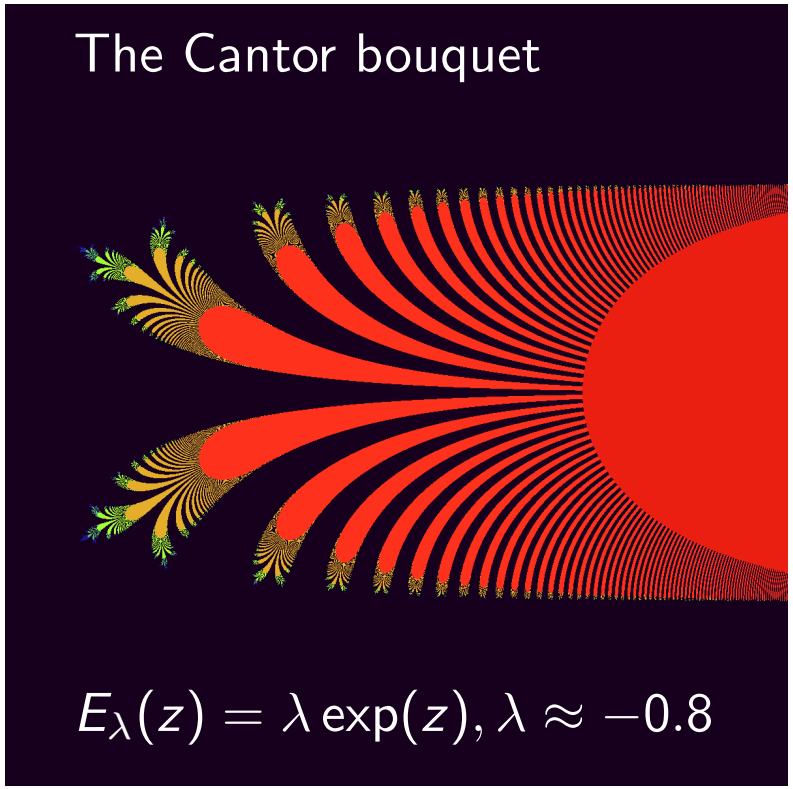
(i) Eremenko's proof using Nevalina Theory (fast escaping set $A(f)$).

(ii) Dominguez's proof using Bohr's Theorem.

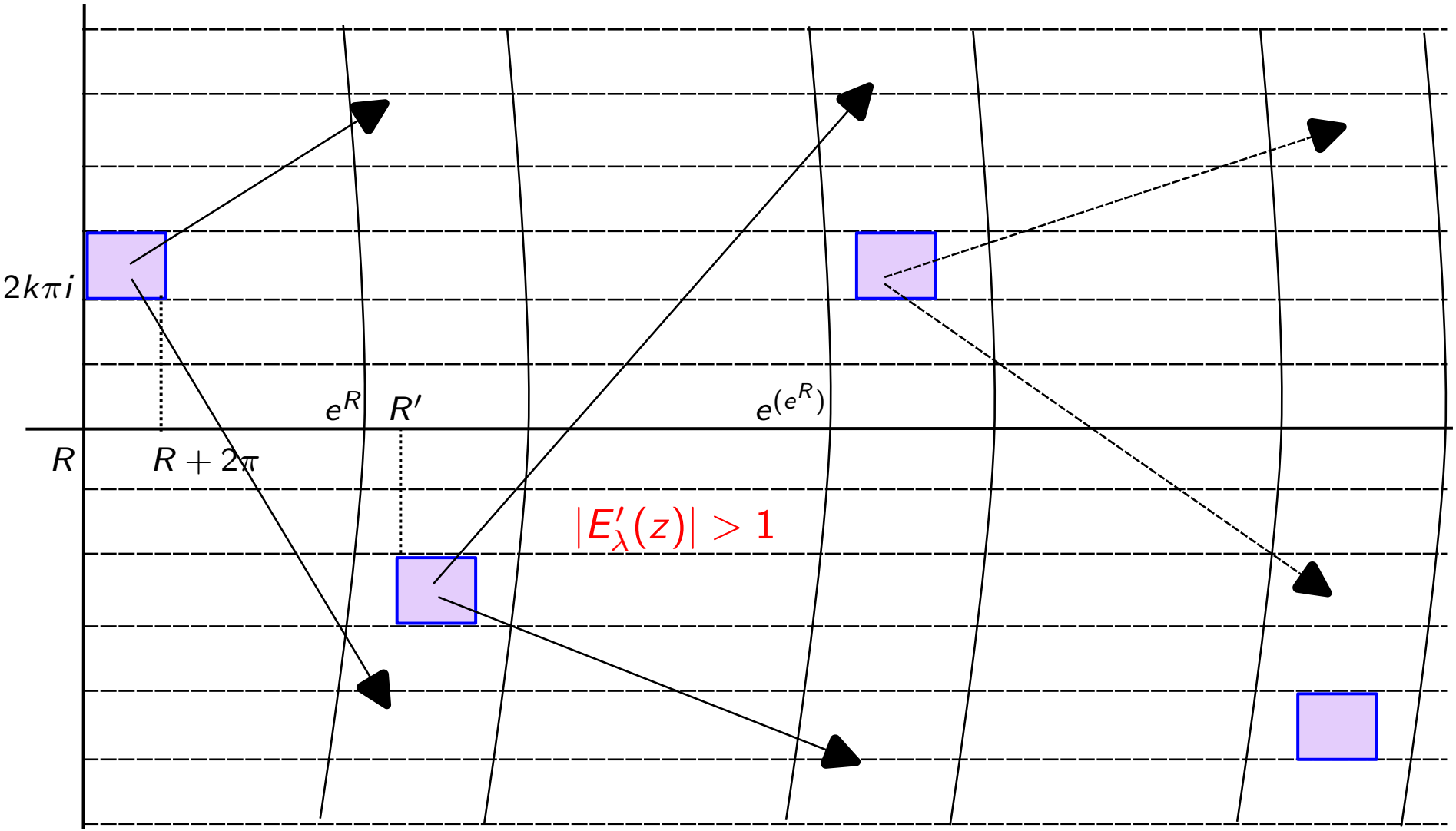
The escaping set for exponentials



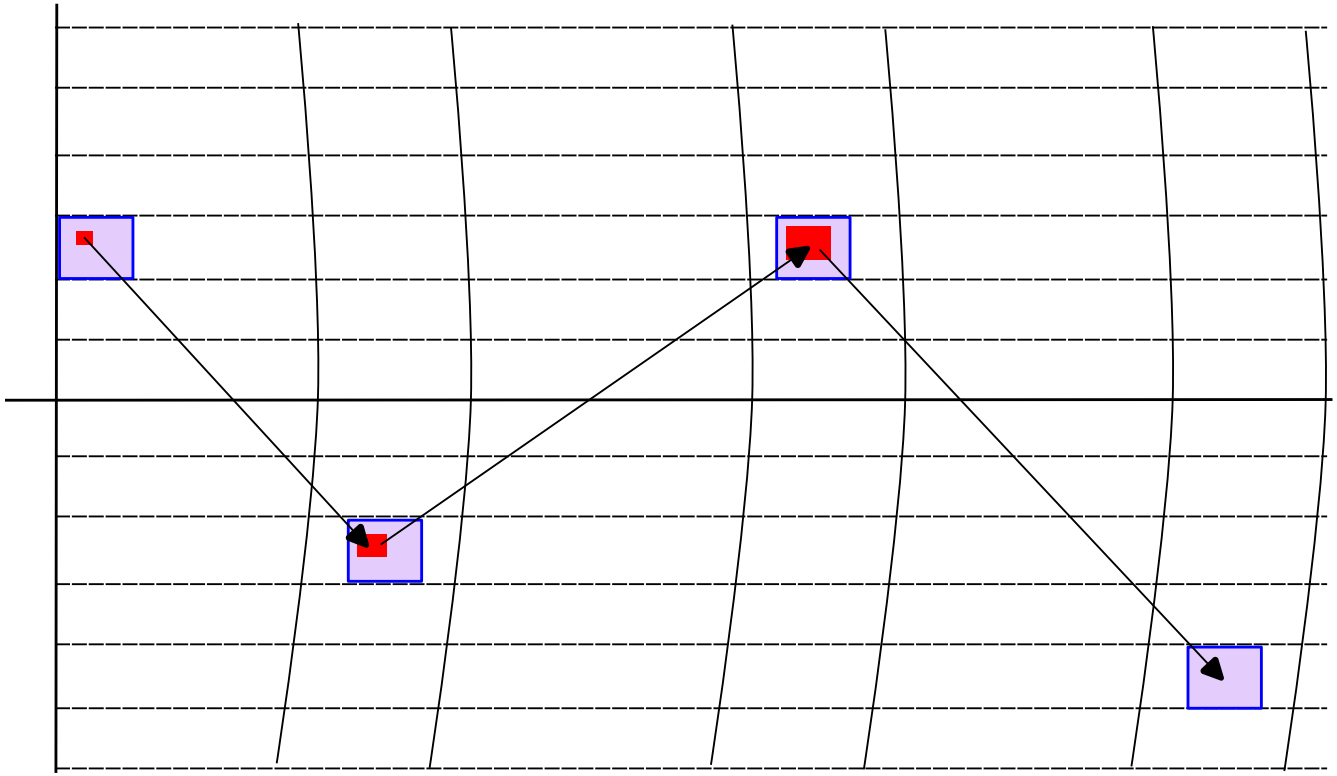
Dynamics of $\exp(z)$ (Devaney-Krych)



Escaping points for exponentials (a toy model)

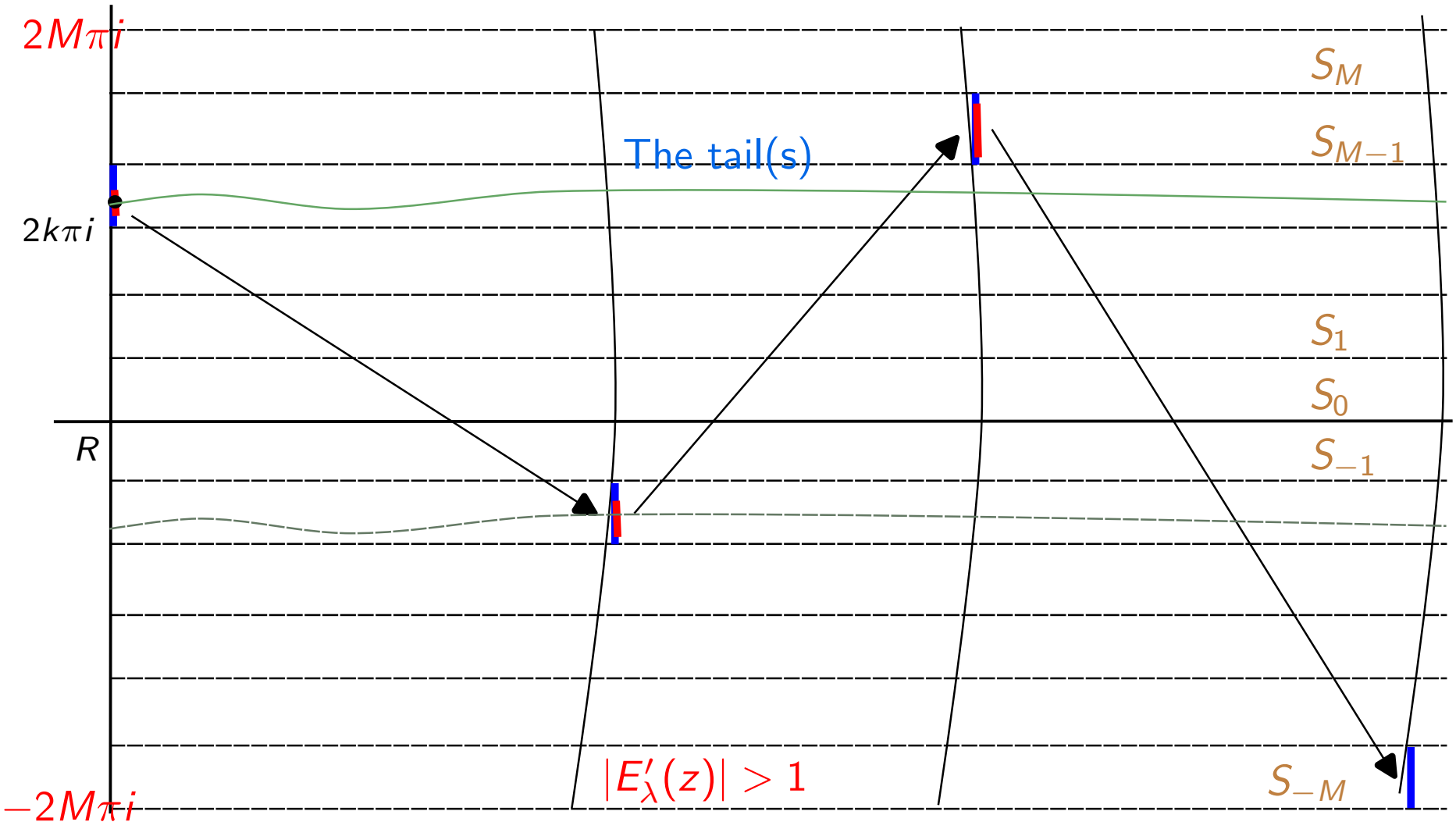


Escaping points for exponentials (a toy model)



Constructing escaping points as limit of the nested process of red rectangles (We use $|E'_\lambda(z)| > 1$ to show that the infinite intersection is just one point).

The tails of the hairs [in $\mathcal{I}(E_\lambda),\ \lambda \in \mathbb{C}^\star$]



The tails of the hairs [in $\mathcal{I}(E_\lambda),\ \lambda \in \mathbb{C}^\star$]

Fix M and $R >> 1$. Let

$$\Sigma_M := \{s = (s_0 s_1 \dots s_n \dots),\ s_j \in \{-M, \dots M\}\}$$

A tail is a parametrized curve $h_s : (R, \infty) \to \mathbb{C}$ such that

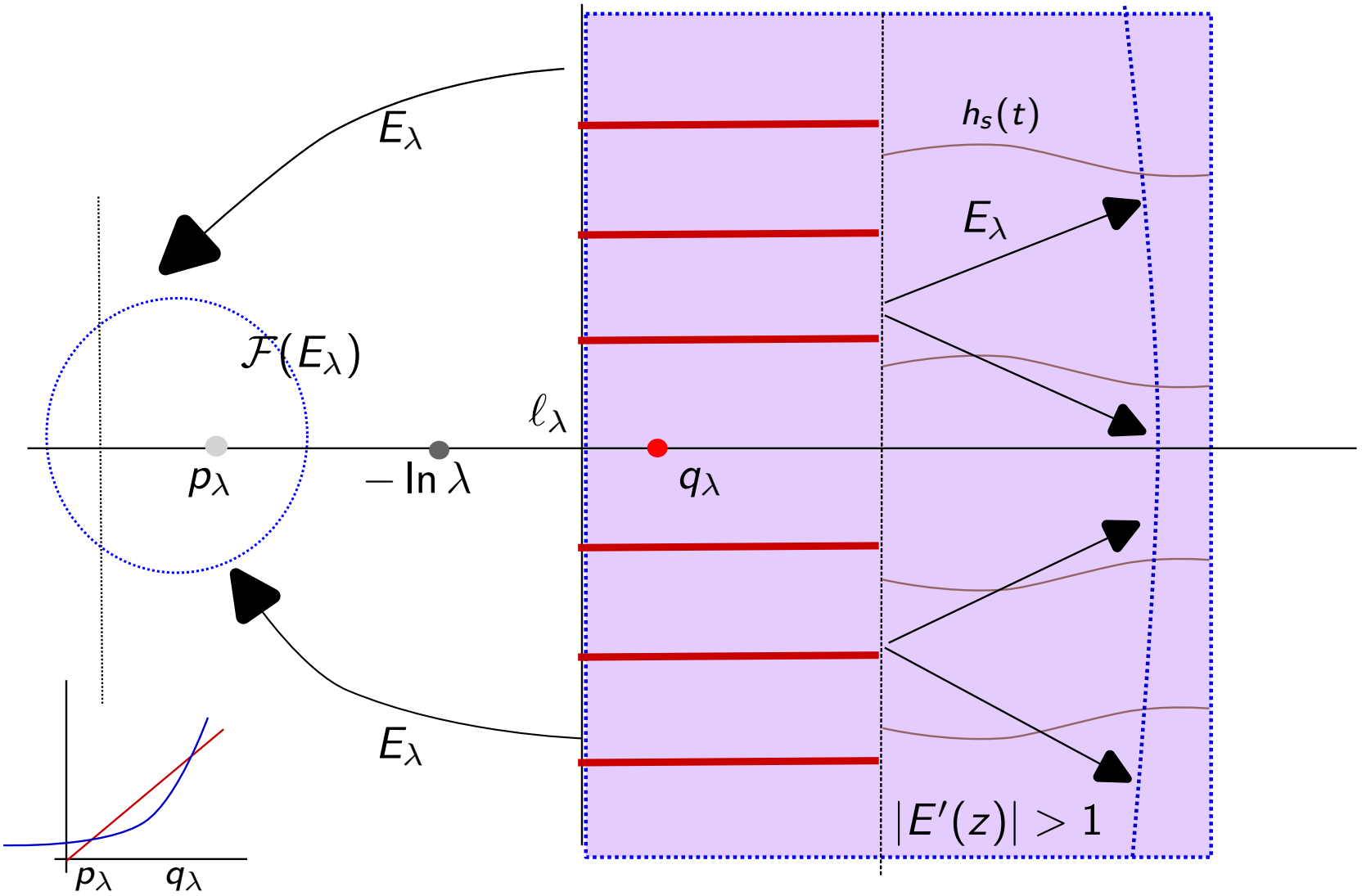
- h_s is a homeomorpishm.
- $\mathrm{Re}(E_\lambda^n(h_s(t))) \to \infty$ as $n \to \infty$. $[h_s \subset \mathcal{I}(E_\lambda),\ s \in \Sigma_M,\ M > 1]$
- For each $t \in (R, \infty)$, $h_s(t)$ lies in the horizontal strip

$$(2s_0 - 1)\pi i < \mathrm{Im}(z) < 2s_0 \pi i \quad [S_{s_0}]$$

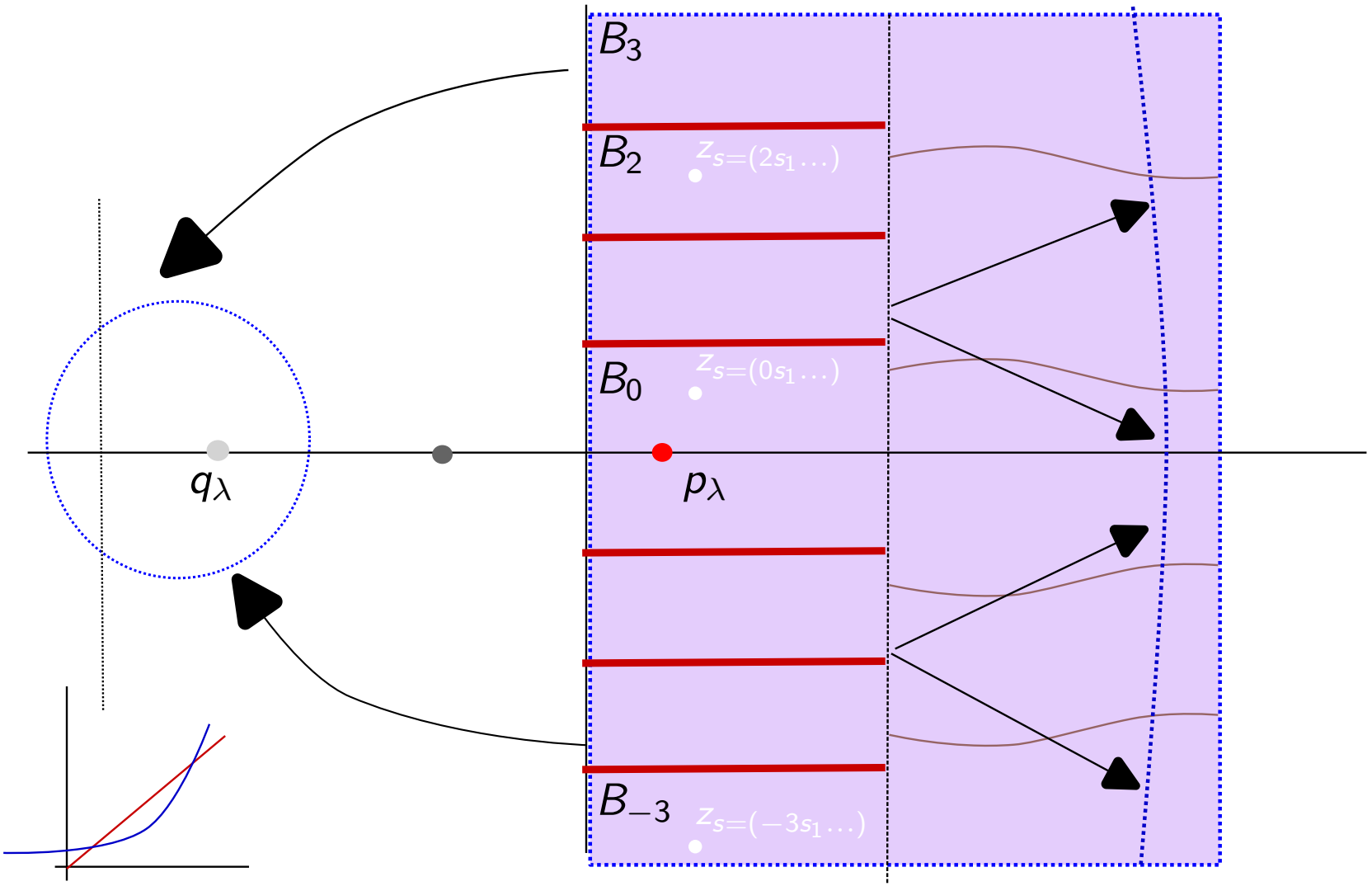
- $E_\lambda(h_s(t)) \subset h_{\sigma(s)}(t)$ where $\sigma : \Sigma_M \to \Sigma_M$ is the shift map

$$\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 \dots).$$

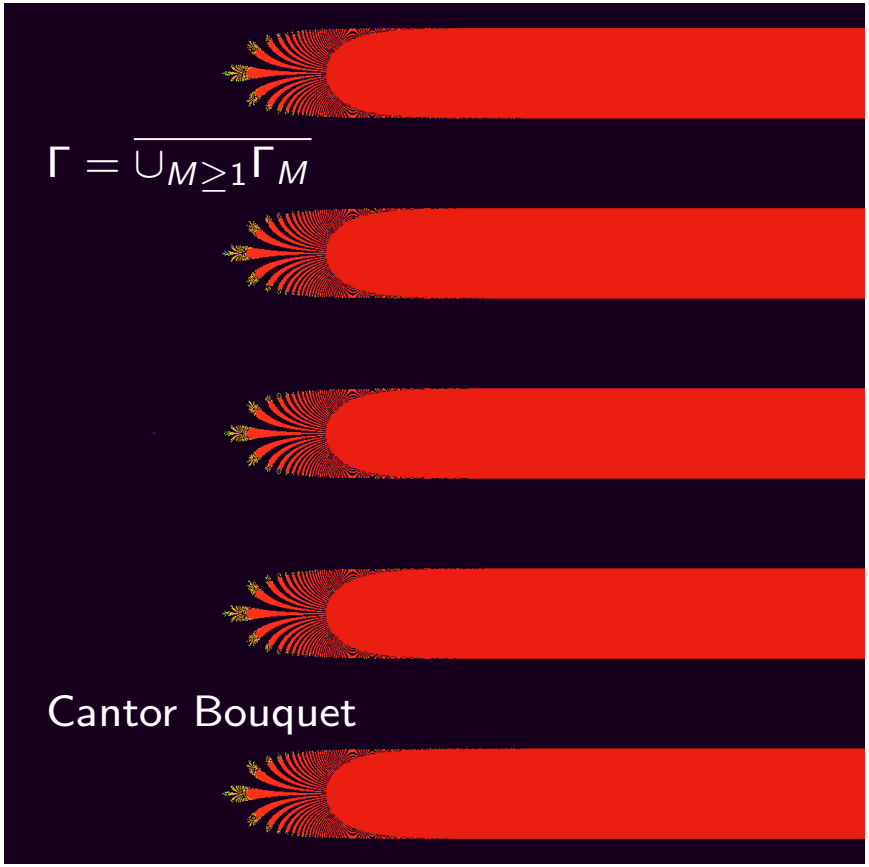
The Cantor bouquet ($E_\lambda,\ \lambda \in (0, 1/e)$)

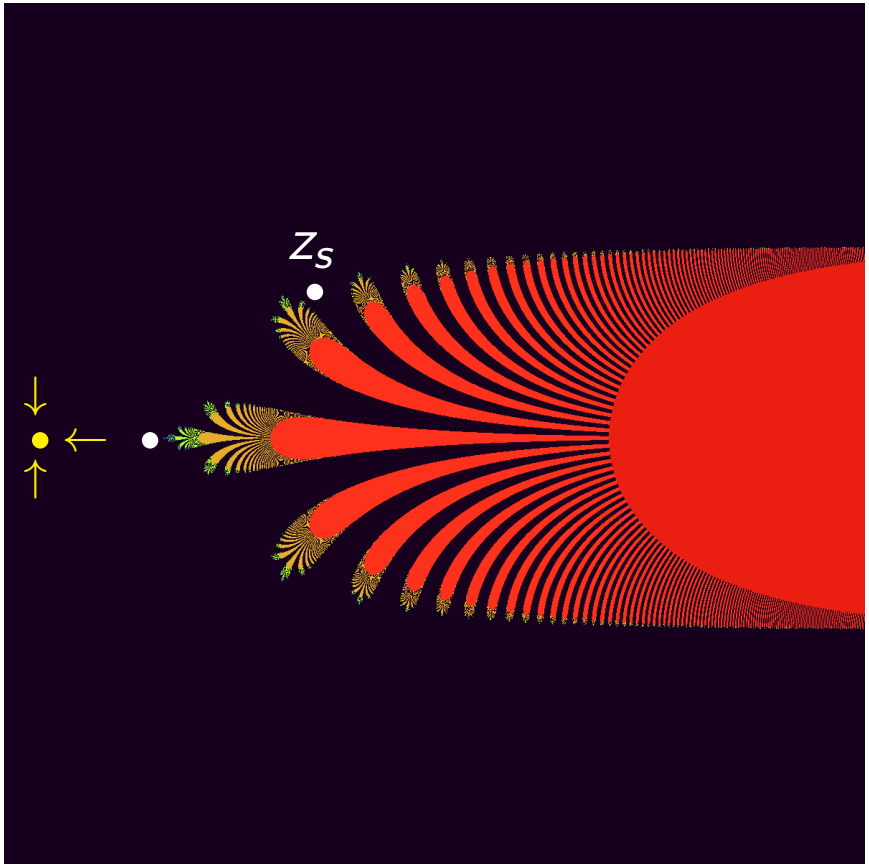


The Cantor bouquet ($E_\lambda,\ \lambda \in (0, 1/e)$)



The Cantor bouquet ($E_\lambda,\ \lambda \in (0, 1/e)$)





The escaping set and the Cantor bouquet

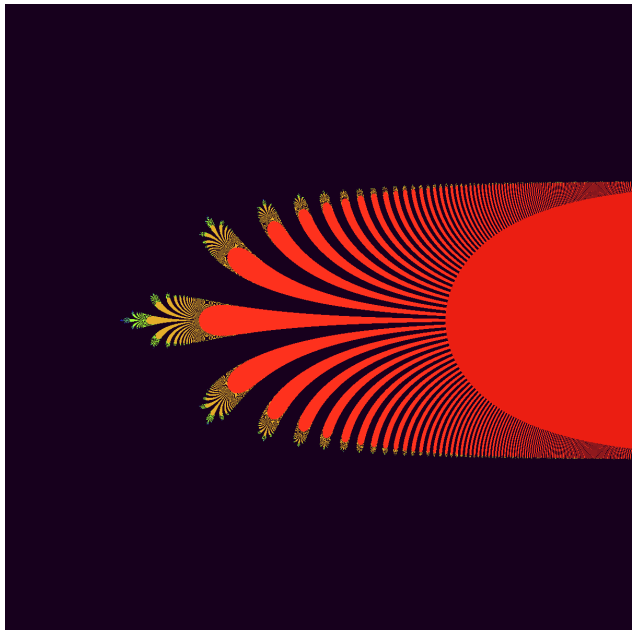
Remark 1. Accordingly, $\mathcal{I}(E_\lambda)$, $\lambda \in (0, 1/e)$, coincides with the union of all hairs (rays) we have constructed and some of the endpoints (white points) corresponding to the ones whose imaginary (and real part) part under iteration tends (slowly) to infinity.

Remark 2. Aarts-Oversteegen showed that there is essentially only one way to embed a Cantor bouquet in the plane (a topological model): The straight brush. Barański (and later Barański-J.-Rempe) extended the Cantor bouquet (straight brush) estructure of the Julia set to a larger family of hyperbolic entire maps.

Remark 3. The tails of the hairs exists for all $\lambda \neq 0$. However for some $\lambda-$parameters ($\lambda = 1$, for instance) we have $\mathcal{J}(E_\lambda) = \mathbb{C}$. In this case not all tails land, but still one can built a Cantor bouquet (and much more invariant structures: indecomposable continua).

Going back to the escaping set: Eremenko's conjecture

Remark. For $E_\lambda, \lambda \in (0, 1/e)$, each connected component of $\mathcal{I}(E_\lambda)$ is a curve (hair) extending to infinity and connecting each point in $\mathcal{I}(E_\lambda)$ with infinity through $\mathcal{I}(E_\lambda)$.



Theorem (Eremenko). Let f be an entire transcendental map. The closure $\overline{\mathcal{I}(f)}$ of $\mathcal{I}(f)$ has no bounded connected components.

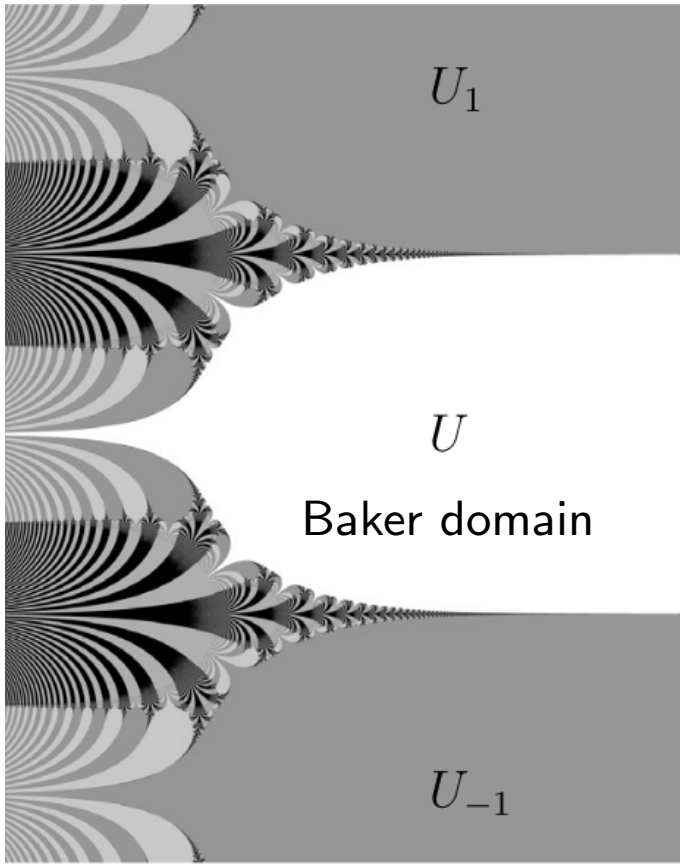
Eremenko's conjecture

Eremenko's Conjecture. The following statement holds.

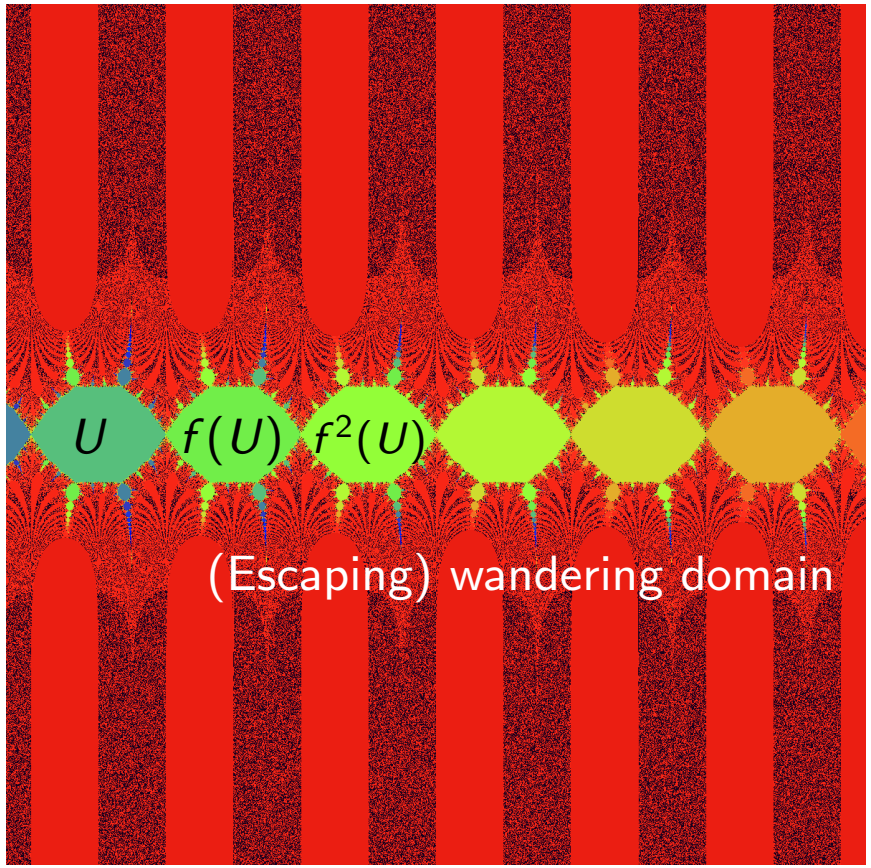
- (It is plausible that) The set $\mathcal{I}(f)$ has no bounded connected components.
- (It is plausible that) The set $\mathcal{I}(f)$ always has the following property: every point $z \in \mathcal{I}(f)$ can be joined with ∞ by a curve in $\mathcal{I}(f)$ (Strong Eremenko's conjecture).

Eremenko's Conjecture. Is the conjecture true? (bids are admitted)

(New) Fatou domains for entire transcendental maps



$f(z) = z + \exp(-z)$



$f(z) = z + 2\pi + \sin(z)$

Theorem (Sullivan, 1981) Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. If $U \in \mathcal{F}(R)$ then it is (pre)-periodic. [Non wandering domains Theorem]

Dynamical classification of wandering domains

Theorem (P. Fatou, 1878-1929). Let U be a wandering domain of f. All limit functions of the (convergence) sequences $\{f^{n_k}|_U\}$ are constant.

- $\{f^n|_U\} \to \infty$ (escaping) $\quad \to \quad U \subset \mathcal{I}(f)$!!
- $\{f^{n_k}|_U\} \to \infty$ and $\{f^{m_k}|_U\} \to a \in \mathcal{J}(f) \subset \mathbb{C}$ (oscillating)
- If $\{f^{n_k}|_U\} \to a$ then $a \neq \infty$ (bounded) $\leftarrow$ dynamically!!!

Theorem (Eremenko-Lyubich (1987) and Bishop (2015)). There exists an entire function f which has an oscillating wandering component U (with infinitely many finite constant limit functions). Such f can be chosen in class $\mathcal{B}$. [Approximation theory + Quasi-conformal folding]

Remark (September 20, 2026): It is not known if there are examples of the third type. Are there? (bids are admitted again)

Eremeko's conjecture: Positive and negative results

Theorem (Schleicher-Zimmer, 2003). Let E_λ the complex exponential family. Then every point in $\mathcal{I}(E_\lambda)$ can be connected to infinity by a curve in $\mathcal{I}(f)$ (strong Eremenko's conjecture holds for exponentials).

Theorem (Rottenfusser-Rückert-Rempe-Schleicher, 2011). Suppose $f \in \mathcal{B}$ (bounded singular set) has finite order. Then the strong Eremenko's conjecture holds.

Theorem (Rottenfusser-Rückert-Rempe-Schleicher, 2011). There exists $f \in \mathcal{B}$ for which the strong Eremenko's conjecture does not hold.

Eremeko's conjecture is false

Theorem (MartíPete-Rempe-Waterman, 2022). There exists a transcendental entire function f such that $z = 0$ is a connected component of $\mathcal{I}(f)$. In particular Eremenko's conjecture fails.

Very much inspired by the following two results.

Theorem (Rippon-Stallard, 2011) Let f be an entire transcendental function. If U is an escaping wandering domain of f, then $\partial U \cap \mathcal{I}(f)$ has full harmonic measure relative to U. If U is an oscillating wandering domain of f, then $\partial U \cap \mathcal{I}(f)$ has zero harmonic measure relative to U.

Theorem (Boc Thaler, 2021) Let $\Omega \in \mathbb{C}$ be a bounded connected regular open set whose closure has a connected complement. There exists an entire function f for which Ω is an escaping wandering domain and the iterates $f^n|_\Omega$ are univalent.

Non existence of bounded wandering domains and AI

A NEW PROOF OF NO-WANDERING DOMAIN THEOREMS WITHOUT QUASICONFORMAL TECHNIQUES

ZIHAO YE

ABSTRACT. In one complex dimension, we use hyperbolic area to prove the absence of wandering Fatou components for rational maps of degree at least two and for transcendental entire functions whose singular sets are compact and have all their accumulation points in the Fatou set. As corollaries, we recover Sullivan's theorem for rational maps and the theorem of Eremenko–Lyubich and Goldberg–Keen for entire functions with finitely many singular values. The proof uses neither quasiconformal deformation nor Teichmüller theory.

Non existence of bounded wandering domains and AI

BOUNDED-ORBIT WANDERING DOMAINS DO NOT EXIST

KOSTIANTYN DRACH, LETICIA PARDO-SIMÓN, AND BENO UČAKAR

ABSTRACT. This paper establishes that transcendental entire functions do not have wandering Fatou components whose forward orbit is bounded. This settles a long-standing question in transcendental dynamics. A stronger statement is proved using the same method: the orbit of every point in a wandering domain of a transcendental *meromorphic* function is unbounded. In particular, transcendental meromorphic functions do not admit *orbitally bounded* wandering domains. The proofs rely mostly on uniform estimates for hyperbolic area, following the approach of Ye [Y].

ABSENCE OF BOUNDED-ORBIT WANDERING DOMAINS

NIKOLAI PROCHOROV, LASSE REMPE, AND JAMES WATERMAN

ABSTRACT. We prove that transcendental entire functions do not have bounded-orbit wandering domains. This answers a long-standing open question. In fact, we prove a more general theorem, establishing the absence of certain wandering domains even for locally defined analytic functions.

The proof is based on a recent remarkable argument of Ye, who obtained an elementary proof of Sullivan's classical theorem that rational maps do not have wandering domains.

Moltes gràcies



Going back to the escaping set: Eremeko's conjecture

Theorem. Let f be a transcendental entire map. The closure $\overline{\mathcal{I}(f)}$ of $\mathcal{I}(f)$ has no bounded connected components.

Proof. By contradiction.

Let $\mathcal{I}_0$ be a connected bounded component of $\overline{\mathcal{I}(f)}$ (observe that $\mathcal{I}_0 \cap \partial\mathcal{I}(f) \neq \emptyset$). Then there exists A a topological annulus separating $\mathcal{I}_0$ from ∞ such that $A \cap \mathcal{I}(f) = \emptyset$. Since $f^n(A) \cap \mathcal{I}(f) = \emptyset$ then $A \subset \mathcal{F}(f)$.

Let K be the bounded connected component of $\mathbb{C} \setminus A$. Since $J(f) = \partial\mathcal{I}(f)$ and $\partial\mathcal{I}(f) \cap \mathcal{I}_0 \neq \emptyset$ we conclude that A is a multiply connected component of $\mathcal{F}(f)$ which implies $A \subset \mathcal{I}(f)$ (maximum principle), a contradiction.

Eremenko's conjecture: Related results

Theorem (Schleicher-Zimmer, 2003). Let E_λ the complex exponential family. Then every point in $\mathcal{I}(E_\lambda)$ can be connected to infinity by a curve in $\mathcal{I}(f)$ (strong Eremenko's conjecture for exponentials).

Theorem (Rempe, 2010). Let $E_1(z) = \exp(z)$. Then $\mathcal{I}(E_1)$ is a connected set.

Theorem (Mihaljević-Brant, 2010). Let $E_1(z) = \lambda \sin(z)$ with λ a Misiurewicz parameter. Then $\mathcal{I}(E_1)$ has infinitely many components.

Theorem (Jarque, 2010). Let $E_\lambda(z) = \lambda \exp(z)$ with λ a Misiurewicz parameter. Then $\mathcal{I}(E_1)$ is a connected set.

Theorem (Rippon-Stallard, 2012) Let f be a transcendental entire function. Then $\mathcal{I}(f)$ has at least one unbounded connected component.